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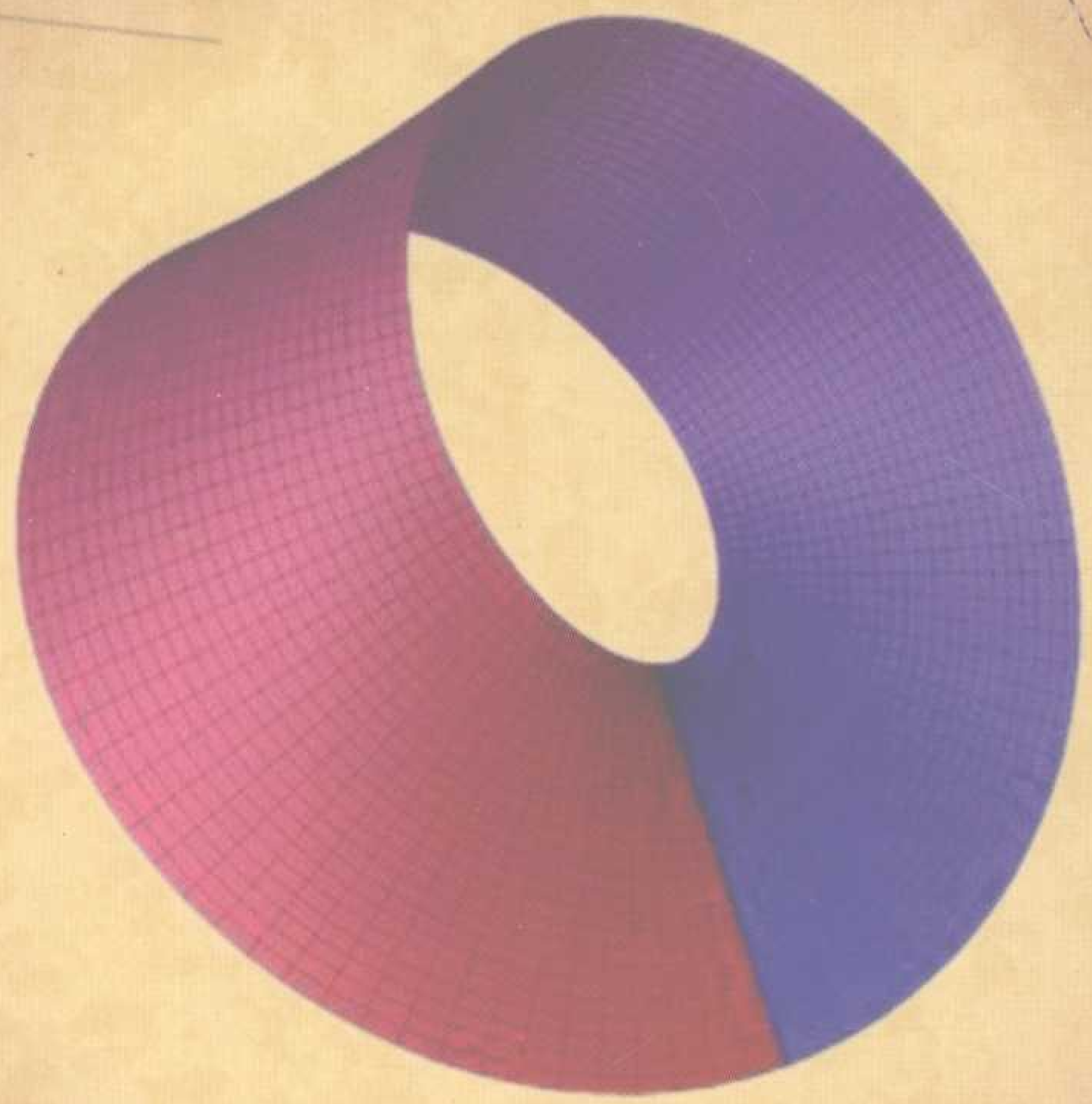

# Topology from the Differentiable Viewpoint

# 从微分观点看拓扑

（双语版）

[美] John W. Milnor 著

熊金城 译

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# Topology from the Differentiable Viewpoint

“时间已经证明，Milnor讲义的价值都是无可估量的。”

——Serge Lang，已故著名数学家

“一本可以让大学三年级学生能看懂的小册子，却包含如此多的深刻的定理（从Sard定理直到Hopf定理）以及完整的证明，这是何等地不可思议！这是学拓扑或几何的学生的必读书。”

——杨劲根教授，复旦大学

本书由菲尔兹和沃尔夫双奖获得者、杰出的数学家John W. Milnor所著，是一部蜚声国际数学界的经典之作。作者从微分同胚和光滑流形等基础概念开始，进而探讨了切空间、定向流形和向量场，讨论了同伦、映射的指数和Pontryagin构造等重要概念，并证明了Sard定理和Hopf定理，简洁而又清晰地介绍了现代数学中的重要专题。

本书前半部分为中译稿，后半部分提供了英文原稿，方便双语教学，也有利于读者深入理解原著。

**John W. Milnor** 著名美国数学家，菲尔兹奖（1962）和沃尔夫奖（1989）得主。美国科学院院士，1966年获得美国国家科学奖章。现任纽约州立大学石溪分校教授。在微分拓扑、K理论、动力系统等方面都有杰出的成就。他的写作风格深受读者欢迎，除本书外，还著有*Morse Theory*、*Characteristic Classes*等，都是公认的数学名著。

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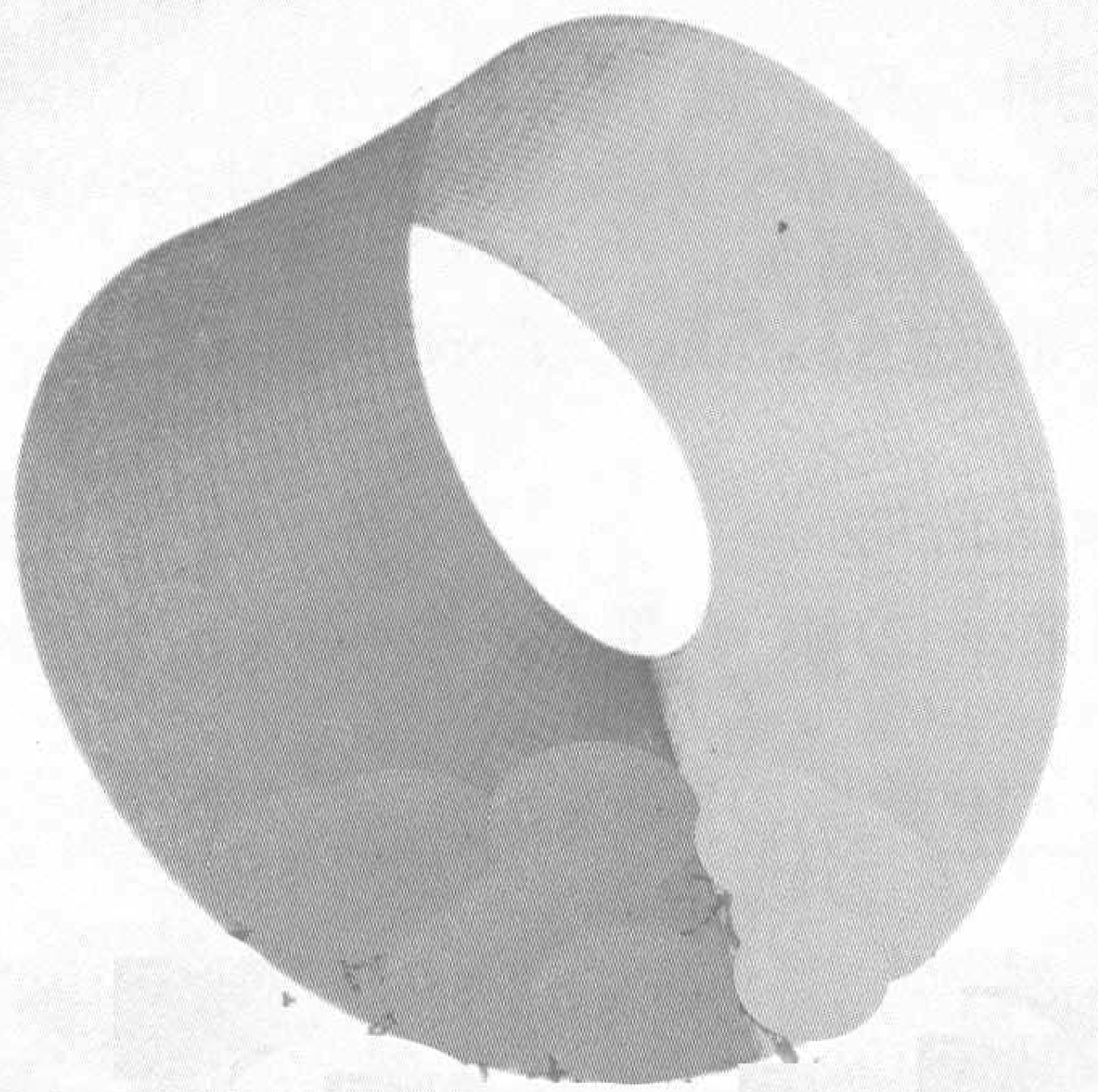

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## 内 容 提 要

本书由菲尔兹奖和沃尔夫奖得主 J. W. Milnor 所著，是一本蜚声国际数学界的经典之作. 内容涉及光滑流形和光滑映射，Sard 定理和 Brown 定理，映射的模 2 度，定向流形，向量场与 Euler 数，标架式协边，Pontryagin 构造等. 全书内容简要，短小精悍.

本书为双语版，可用于双语教学. 既适合高等院校数学专业高年级本科生和研究生阅读，也可供对微分拓扑有兴趣的专业人士参考.

图灵数学·统计学丛书

**从微分观点看拓扑(双语版)**

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# 译 者 介 绍

**熊金城**：男，1938 年出生，江西南昌人. 1962 年毕业于北京大学数学力学系数学专业，先后在中国科学院数学研究所、中国科学技术大学、华南师范大学工作. 曾任中国科学技术大学数学系副主任、中国科学院数学专家委员会委员，以及国际理论物理中心 (意大利) 协约成员. 1992 年获国务院政府特殊津贴. 编有《点集拓扑讲义》(曾获得国家教委大学优秀教材二等奖，该书第 4 版已列入“十一五”国家教材规划). 译著有《从微分观点看拓扑》、《代数拓扑学基础教程》等. 曾在世界各地作学术访问或参加学术会议，是我国微分拓扑领域的著名学者.

# 译 者 序

《从微分观点看拓扑》一书为 1962 年菲尔兹奖和 1989 年沃尔夫奖得主 J. W. Milnor 所著，是一本蜚声国际数学界的经典之作. 在这本书中, 作者用微分拓扑的方法去处理拓扑学中的一些典型论题. 通过如此短小的篇幅, 如此快捷地向读者展现拓扑学中引人入胜的成果, 不仅展现出了作者的深厚功力, 同时也展现出了微分拓扑方法的巨大功效.

阅读这本书, 并不需要许多数学知识作为基础, 大学数学系二、三年级以上的学生, 不会感到困难. 我认为所有的数学工作者, 无论他们是否打算从事几何拓扑方向的研究工作, 都将因阅读此书而从中获益.

大约是在 1965 年末或 1966 年初这本书刚出版不久的时候, 我便有幸读到了它. 那时我大学毕业不久, 是个年轻人, 在中国科学院数学研究所拓扑组学习做微分拓扑奇点理论方面的研究工作. 这本十分精巧的小书当年令我爱不释手, 给我留下了极为深刻的印象. 然而第一次把它翻译成中文, 却是在 10 余年后的 1978 年, 我已届中年了. 当时南京师范学院 (早已改名为南京师范大学了) 数学系邀请我去他们那里就拓扑学的有关问题作一些交流, 在南京待了将近两个月. 在这过程中利用空余的时间, 完成了翻译工作, 后经由上海科学技术出版社出版. 这便是该书的第一个中文译本的来由. 这次, 应人民邮电出版社图灵公司之邀重译, 我已是垂垂老矣. 此书可说是与我结下了不解之缘. 这次重译, 改正了原译本中的某些不妥之处. 由于译者水平有限, 疏漏和不当之处恐怕还是难免, 尚祈读者不吝指正.

译者

2008 年春于华南师范大学

纪念 Heinz Hopf

# 序

本书源自 1963 年 12 月我在佩奇–巴伯 (Page-Barbour) 讲义基金会的资助下于弗吉尼亚 (Virginia) 大学所做讲座的讲义. 其中介绍了拓扑学由来已久的某些论题, 这些论题是以 1912 年 L. E. J. Brouwer 的映射度的定义为中心的. 然而, 我们用的是微分拓扑的方法而不是 Brouwer 的组合方法. 正则值的概念以及断定每一个光滑映射都有正则值的 Sard 定理和 Brown 定理在本书中起着核心作用.

为方便陈述起见, 所有的流形都取为无限次可微的并且明确地嵌入在欧氏空间中的流形. 并假定读者具备点集拓扑学和实变理论方面的一些知识.

在此我想向 David Weaver 表示我的谢意, 他的早逝使我们大家都深为悲痛. 本书初稿全赖他出色的笔记才得以问世.

J. W. M.

1965 年 3 月于新泽西州普林斯顿

# PREFACE

These lectures were delivered at the University of Virginia in December 1963 under the sponsorship of the Page-Barbour Lecture Foundation. They present some topics from the beginnings of topology, centering about L. E. J. Brouwer's definition, in 1912, of the *degree* of a mapping. The methods used, however, are those of differential topology, rather than the combinatorial methods of Brouwer. The concept of *regular value* and the theorem of Sard and Brown, which asserts that every smooth mapping has regular values, play a central role.

To simplify the presentation, all manifolds are taken to be infinitely differentiable and to be explicitly embedded in euclidean space. A small amount of point-set topology and of real variable theory is taken for granted.

I would like here to express my gratitude to David Weaver, whose untimely death has saddened us all. His excellent set of notes made this manuscript possible.

J. W. M.

*Princeton, New Jersey*
*March 1965*

# 目　录

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# 第 1 章　光滑流形和光滑映射

首先解释一些术语. $\mathbf{R}^k$ 表示 $k$ 维欧氏空间, 于是一个点 $x \in \mathbf{R}^k$ 便是实数的一个 $k$ 重组 $x = (x_1, \cdots, x_k)$.

设 $U \subset \mathbf{R}^k$, $V \subset \mathbf{R}^l$ 都是开集. 如果从 $U$ 到 $V$ 的映射 $f$ (写作 $f: U \to V$) 的所有偏导数 $\partial^n f/\partial x_{i_1} \cdots \partial x_{i_n}$ 都存在且连续, 则称 $f$ 为一个*光滑映射* smooth map.

更为一般的情形是, 设 $X \subset \mathbf{R}^k$ 以及 $Y \subset \mathbf{R}^l$ 为欧氏空间的任意子集, $f: X \to Y$. 如果对于每一个 $x \in X$, 存在着包含 $x$ 的开集 $U \subset \mathbf{R}^k$ 以及光滑映射 $F: U \to \mathbf{R}^l$ 使得 $F$ 与 $f$ 在 $U \cap X$ 上是一致的, 则称 $f$ 为一个*光滑映射*.

如果 $f: X \to Y$ 与 $g: Y \to Z$ 都是光滑的, 那么复合映射 $g \circ f: X \to Z$ 也是光滑的. 任何一个集合 $X$ 的恒同映射显然是光滑的.

**定义**　如果映射 $f: X \to Y$ 将 $X$ 同胚地变到 $Y$ 上, 并且 $f$ 与 $f^{-1}$ 两者都是光滑的, 则称 $f$ 为一个*微分同胚* (diffeomorphism).

现在可以粗略地说, *微分拓扑学* (differential topology) 研究的是一个集合 $X \subset \mathbf{R}^k$ 在微分同胚下不变的性质.

我们不打算去考察那些完全任意的集合 $X$, 而是通过下述定义挑选出特别引人注目和特别有用的一类集合.

**定义**　设 $M \subset \mathbf{R}^k$. 如果对于每一个点 $x \in M$, 都有一个邻域 $W \cap M$ 微分同胚于欧氏空间 $\mathbf{R}^m$ 的某一个开子集 $U$, 则称 $M$ 为一个 $m$ 维*光滑流形* (smooth manifold).

任何一个特定的微分同胚 $g: U \to W \cap M$ 称为区域 $W \cap M$ 的一个*参数化* (parametrization). [逆微分同胚 $g^{-1}: W \cap M \to U$ 称为 $W \cap M$ 上的一个*坐标系* (system of coordinates).]

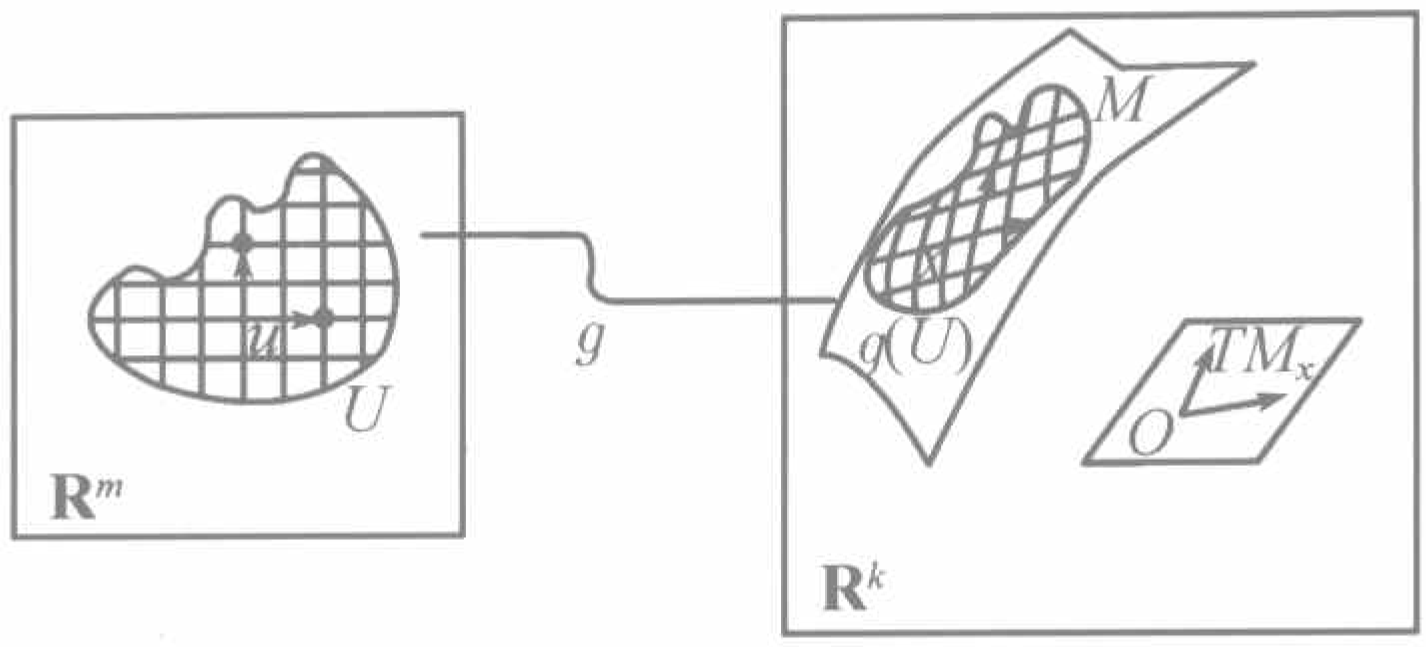


图 1　$M$ 中区域的参数化

有时, 我们必须考察零维流形. 根据定义, 如果对于每一个 $x \in M$, 邻域 $W \cap M$ 都由 $x$ 独点组成, 则 $M$ 是一个零维流形.

**例**　由所有满足 $x^2 + y^2 + z^2 = 1$ 的点 $(x, y, z) \in \mathbf{R}^3$ 组成的单位球 $S^2$ 是 2 维光滑流形. 事实上, 当 $x^2 + y^2 < 1$ 时, 微分同胚

$$(x, y) \to (x, y, \sqrt{1 - x^2 - y^2})$$

将 $S^2$ 中 $z > 0$ 的区域参数化了. 用互换 $x$, $y$, $z$ 以及改变变量符号的办法, 得到 $x > 0$, $y > 0$, $x < 0$, $y < 0$ 以及 $z < 0$ 的各个区域类似的参数化. 由于这些区域覆盖 $S^2$, 所以 $S^2$ 是一个光滑流形.

更为一般的情形是, 由所有满足方程 $\Sigma x_i^2 = 1$ 的点 $(x_1, \cdots, x_n)$ 组成的球面$S^{n-1} \subset \mathbf{R}^n$ 是一个 $n-1$ 维光滑流形. 例如 $S^0 \subset \mathbf{R}^1$ 是由两个点组成的流形.

一个不大规整的光滑流形的例子可由所有满足条件 $x \neq 0$ 和 $y = \sin(1/x)$ 的点 $(x, y) \in \mathbf{R}^2$ 的集合给出.

## 1.1　切空间和导射

为了对光滑流形间的光滑映射 $f : M \to N$ 定义 **导射** (derivative) $\mathrm{d}f_x$ 的概念, 首先将每一个点 $x \in M \subset \mathbf{R}^k$ 联系一个 $m$ 维线性子空间 $TM_k \subset \mathbf{R}^k$, 并称之为 $M$ 在点 $x$ 处的 **切空间** (tangent space). 然后, $\mathrm{d}f_x$ 将是一个从 $TM_x$ 到 $TN_y$ 的线性映射, 其中 $y = f(x)$. 向量空间 $TM_x$ 的元素称为 $M$ 在点 $x$ 处的 **切向量** (tangent vector).

人们直观地联想到 $\mathbf{R}^k$ 中的在点 $x$ 附近最好地逼近 $M$ 的 $m$ 维超平面, $TM_x$ 便是既通过原点而又平行于上述超平面的那个超平面. (参

见图 1 和图 2.) 类似地, 人们联想到从点 $x$ 处的切超平面到点 $y$ 处的切超平面的, 最好地逼近 $f$ 的非齐次线性映射. 把这两个超平面都平移到原点去, 便得到 $\mathrm{d}f_x$.

在给出实际定义之前, 必须研究开集间的映射这一特殊情形. 对于任意开集 $U \subset \mathbf{R}^k$, 切空间 $TU_x$ 定义为整个向量空间 $\mathbf{R}^k$. 对于任何一个光滑映射 $f: U \to V$, 导射

$$\mathrm{d}f_x : \mathbf{R}^k \longrightarrow \mathbf{R}^l$$

由下列公式定义：当 $x \in U$, $h \in \mathbf{R}^k$,

$$\mathrm{d}f_x(h) = \lim_{t \to 0}(f(x+th) - f(x))/t.$$

显然, $\mathrm{d}f_x(h)$ 是 $h$ 的线性函数. [实际上, $\mathrm{d}f_x$ 恰好是与在点 $x$ 处取值的一阶偏导数的 $l \times k$ 阶矩阵 $(\partial f_i/\partial x_j)_x$ 对应的线性映射.]

下面是导射运算的两条基本性质:

(1) (链法则) 若 $f: U \to V$ 和 $g: V \to W$ 都是光滑映射, $f(x) = y$, 则

$$\mathrm{d}(g \circ f)_x = \mathrm{d}g_y \circ \mathrm{d}f_x.$$

换言之, $\mathbf{R}^k$、$\mathbf{R}^l$、$\mathbf{R}^m$ 的开子集之间的光滑映射的每一个可交换的三角形

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "g\circ f"] & & W \end{tikzcd}$$

都对应着一个线性映射的可交换的三角形

$$\begin{tikzcd} & \mathbf{R}^l \arrow[dr, "\mathrm{d}g_y"] & \\ \mathbf{R}^k \arrow[ur, "\mathrm{d}f_x"] \arrow[rr, "\mathrm{d}(g\circ f)_x"] & & \mathbf{R}^m. \end{tikzcd}$$

(2) 若 $I$ 为 $U$ 的恒同映射, 则 $\mathrm{d}I_x$ 为 $\mathbf{R}^k$ 的恒同映射. 更为一般的情况是: 如果 $U \subset U'$ 都是开集, 并且

$$i: U \longrightarrow U'$$

为包含映射, 则 $dI_x$ 也是 $\mathbf{R}^k$ 的恒同映射.

还要注意:

(3) 若 $L:\mathbf{R}^k \to \mathbf{R}^l$ 为线性映射, 则 $dL_x = L$.

作为这两条性质的简单应用, 有下述命题:

**命题**　若 $f$ 是开集 $U\subset\mathbf{R}^k$ 与 $V\subset\mathbf{R}^l$ 之间的一个微分同胚, 则 $k$ 必定等于 $l$, 并且线性映射

$$df_x:\mathbf{R}^k\longrightarrow\mathbf{R}^l$$

必定是非退化的.

**证明**　复合映射 $f^{-1}\circ f$ 是 $U$ 的恒同映射, 因此 $d(f^{-1})_y\circ df_x$ 是 $\mathbf{R}^k$ 的恒同映射. 类似地, $df_x\circ d(f^{-1})_y$ 是 $\mathbf{R}^l$ 的恒同映射. 于是 $df_x$ 有双边的逆, 从而推得 $k=l$.

这一命题的部分逆命题为真. 令 $f:U\to\mathbf{R}^k$ 为光滑映射, 其中 $U$ 为 $\mathbf{R}^k$ 中的开集.

**反函数定理**　如果导射 $df_x:\mathbf{R}^k\to\mathbf{R}^k$ 是非退化的, 则 $f$ 将围绕 $x$ 的任何一个充分小的开集 $U'$ 微分同胚地映射到开集 $f(U')$ 上.

(见 Apostol [2, p.144] 或 Dieudonne [7, p.268].)

注意: 即使在每一个点处, $df_x$ 都是非退化的, 但一般而言 $f$ 却可以不是一一映射. (一个有启发性的例子是复平面到自身的指数映射.)

现在对任意的光滑流形 $M\subset\mathbf{R}^k$ 定义切空间 $TM_x$ 如下. 选取 $M$ 中 $x$ 的邻域 $g(U)$ 的一个参数化

$$g:U\longrightarrow M\subset\mathbf{R}^k,$$

其中 $g(u)=x$. 此处 $U$ 是 $\mathbf{R}^m$ 的一个开子集. 将 $g$ 认作是从 $U$ 到 $\mathbf{R}^k$ 的映射. 所以导射

$$dg_u:\mathbf{R}^m\longrightarrow\mathbf{R}^k$$

已有定义. 令 $TM_x$ 等于 $dg_u$ 的象 $dg_u(\mathbf{R}^m)$. (参见图 1.)

必须证明这种构造法不依赖于参数化 $g$ 的特殊选取. 令 $h:V\to M\subset\mathbf{R}^k$ 为 $M$ 中的点 $x$ 的邻域 $h(V)$ 的另外一个参数化, 且令 $v=h^{-1}(x)$. 则 $h^{-1}\circ g$ 将 $u$ 的某一邻域 $U_1$ 微分同胚地映射到 $v$ 的一个邻

域 $V_1$ 上. 开集间的光滑映射的交换图

$$\begin{tikzcd} & \mathbf{R}^k & \\ U_1 \arrow[ur, "g"] \arrow[rr, "h^{-1}\circ g"] & & V_1 \arrow[ul, "h"'] \end{tikzcd}$$

引出线性映射的交换图

$$\begin{tikzcd} & \mathbf{R}^k & \\ \mathbf{R}^m \arrow[ur, "\mathrm{d}g_u"] \arrow[rr, "\cong", "\mathrm{d}(h^{-1}\circ g)_u"'] & & \mathbf{R}^m, \arrow[ul, "\mathrm{d}h_y"'] \end{tikzcd}$$

而由此直接推得 $\mathrm{d}g_u$ 的象等于 $\mathrm{d}h_v$ 的象, 即

$$\mathrm{Image}(\mathrm{d}g_u) = \mathrm{Image}(\mathrm{d}h_v).$$

于是, $TM_x$ 是完全确定的.

**证明 $TM_x$ 是 $m$ 维向量空间.** 因为

$$g^{-1} : g(U) \longrightarrow U$$

是光滑映射, 所以可以选取一个包含 $x$ 的开集 $W$ 以及一个光滑映射 $F : W \to \mathbf{R}^m$ 使 $F$ 在 $W \cap g(U)$ 上与 $g^{-1}$ 一致. 令 $U_0 = g^{-1}(W \cap g(U))$, 可得交换图

$$\begin{tikzcd} & W \arrow[dr, "F"] & \\ U_0 \arrow[ur, "g"] \arrow[rr, "\text{包含映射}"] & & \mathbf{R}^m, \end{tikzcd}$$

因此有交换图

$$\begin{tikzcd} & \mathbf{R}^k \arrow[dr, "\mathrm{d}F_x"] & \\ \mathbf{R}^m \arrow[ur, "\mathrm{d}g_u"] \arrow[rr, "\text{恒同映射}"] & & \mathbf{R}^m. \end{tikzcd}$$

该图显然蕴含 $\mathrm{d}g_u$ 秩为 $m$, 因而它的象 $TM_x$ 为 $m$ 维.

现在考虑两个光滑流形 $M \subset \mathbf{R}^k$ 和 $N \subset \mathbf{R}^l$ 以及光滑映射

$$f : M \longrightarrow N,$$

并设 $f(x) = y$. 导射

$$\mathrm{d}f_x : TM_x \longrightarrow TN_y$$

定义如下. 由于 $f$ 是光滑的, 所以存在着包含 $x$ 的开集 $W$ 以及在 $W \cap M$ 上与 $f$ 一致的光滑映射

$$F : W \longrightarrow \mathbf{R}^l.$$

对于所有 $v \in TM_x$, 定义 $\mathrm{d}f_x(v)$ 等于 $\mathrm{d}F_x(v)$.

为了验证这一定义, 必须证明 $\mathrm{d}F_x(v)$ 属于 $TN_y$ 并且不依赖于 $F$ 的特殊选取.

对于 $x$ 的邻域 $g(U)$ 和 $y$ 的邻域 $h(V)$, 选取参数化

$$g : U \longrightarrow M \subset \mathbf{R}^k \quad \text{和} \quad h : V \longrightarrow N \subset \mathbf{R}^l.$$

如果必要的话, 用一个较小的集合替换 $U$, 则可以假定 $g(U) \subset W$, 以及 $f$ 将 $g(U)$ 映射到 $h(V)$ 中, 从而

$$h^{-1} \circ f \circ g : U \longrightarrow V$$

是一个完全确定的光滑映射.

考虑开集间的光滑映射的交换图

$$\begin{tikzcd} W \arrow[r, "F"] & \mathbf{R}^l \\ U \arrow[u, "g"] \arrow[r, "h^{-1}\circ f\circ g"] & V \arrow[u, "h"'] \end{tikzcd}.$$

取导射, 得到线性映射的交换图

$$\begin{tikzcd} \mathbf{R}^k \arrow[r, "\mathrm{d}F_x"] & \mathbf{R}^l \\ \mathbf{R}^m \arrow[u, "\mathrm{d}g_u"] \arrow[r, "\mathrm{d}(h^{-1}\circ f\circ g)_u"] & \mathbf{R}^n \arrow[u, "\mathrm{d}h_v"'] \end{tikzcd},$$

其中 $u = g^{-1}(x)$, $v = h^{-1}(y)$.

可立即推出, $\mathrm{d}F_x$ 将 $TM_x = \mathrm{Image}(\mathrm{d}g_u)$ 变到

$$TN_y = \mathrm{Image}(\mathrm{d}h_v)$$

中. 从而得到的映射 $\mathrm{d}f_x$ 不依赖于 $F$ 的特殊选取, 因为绕着图表的底部走能得到同一个线性变换. 此即

$$\mathrm{d}f_x = \mathrm{d}h_v \circ \mathrm{d}(h^{-1} \circ f \circ g)_u \circ (\mathrm{d}g_u)^{-1},$$

这就证明了

$$\mathrm{d}f_x : TM_x \longrightarrow TN_y$$

是完全确定的线性映射.

像前面一样, 导射运算具有两条基本性质.

(1) (链法则) *若 $f: M \to N$ 以及 $g: N \to P$ 都是光滑的, $f(x) = y$, 则*

$$\mathrm{d}(g \circ f)_x = \mathrm{d}g_y \circ \mathrm{d}f_x.$$

(2) *若 $I$ 是 $M$ 的恒同映射, 则 $\mathrm{d}I_x$ 是 $TM_x$ 的恒同映射. 更一般些, 如果 $M \subset N$, $i$ 是包含映射, 则 $TM_x \subset TN_x$, $\mathrm{d}i_x$ 也是包含映射.* (参见图 2.)

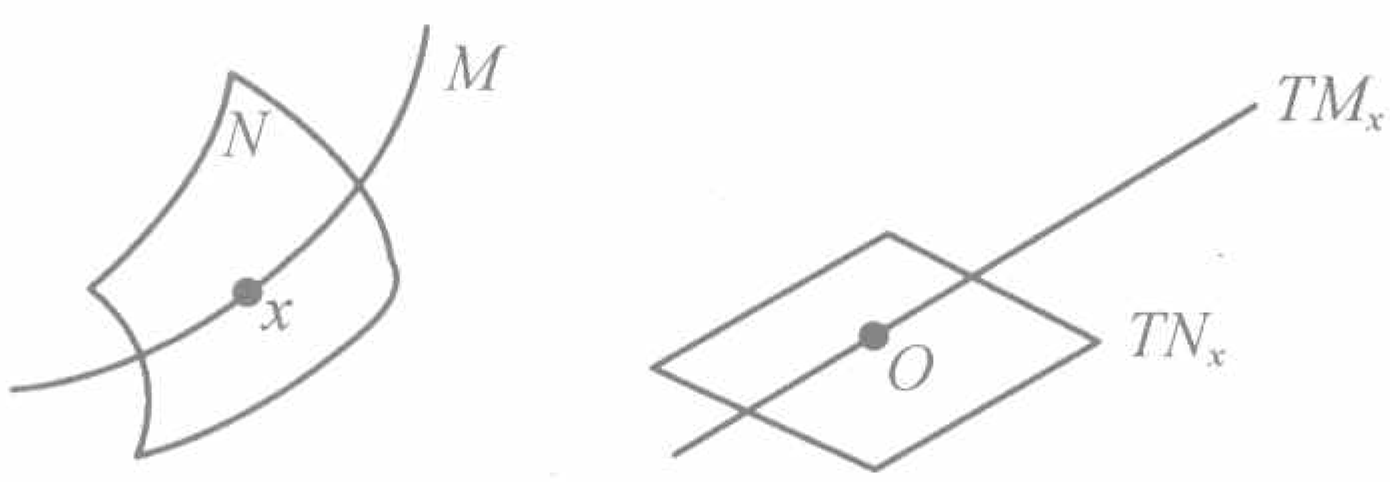


图 2 子流形的切空间

证明是简易的.

像前面一样, 由这两条性质可导出如下命题:

**命题** 若 $f: M \to N$ 是一个微分同胚, 则 $\mathrm{d}f_x : TM_x \to TN_y$ 是向量空间的一个同构. 特别地, $M$ 的维数必定等于 $N$ 的维数.

## 1.2 正 则 值

设 $f : M \to N$ 为维数相同的[①]两个流形之间的一个光滑映射.

① 在第 2 章中这一限制将被取消.

如果在 $x \in M$ 处导射 $\mathrm{d}f_x$ 是非退化的, 则称 $x$ 是 $f$ 的一个*正则点* (regular point). 这时, 从反函数定理推出 $f$ 将 $M$ 中 $x$ 的某一邻域微分同胚地映射到 $N$ 中的某一个开集上. 对于 $y \in N$, 如果 $f^{-1}(y)$ 只包含正则点, 那么 $y$ 称为一个*正则值* (regular value).

若 $\mathrm{d}f_x$ 是退化的, 则 $x$ 称为 $f$ 的一个*临界点* (critical point), 并且象 $f(x)$ 称为一个*临界值* (critical value). 于是每一个点 $y \in N$ 是临界值还是正则值, 则可按照 $f^{-1}(y)$ 包含还是不包含临界点而确定.

*注意: 若 $M$ 是紧致的, 并且 $y \in N$ 是一个正则值, 则 $f^{-1}(y)$ 是一个有限集 (可能是空集).* 这是因为: 一方面, 在任何情况下, $f^{-1}(y)$ 作为紧致空间 $M$ 的闭子集总是紧致的; 另一方面, 由于 $f$ 在每一个点 $x \in f^{-1}(y)$ 的某一邻域中是一一的, 因而 $f^{-1}(y)$ 是离散的.

对于光滑映射 $f: M \to N$, 其中 $M$ 是紧致的, $y \in N$ 是一个正则值, *定义 $\#f^{-1}(y)$ 为 $f^{-1}(y)$ 中点的个数*. 首先注意: $\#f^{-1}(y)$ 作为 $y$ 的函数 (其中 $y$ 只取正则值!) 是局部常值的, 即*存在着 $y$ 的一个邻域 $V \subset N$, 使得对于任意 $y' \in V$ 都有 $\#f^{-1}(y') = \#f^{-1}(y)$*. [设 $x_1, \cdots, x_k$ 为 $f^{-1}(y)$ 的所有点, 选取这些点的两两无交的邻域 $U_1, \cdots, U_k$, 要求这些邻域分别被 $f$ 微分同胚地映射到 $N$ 中某些邻域 $V_1, \cdots, V_k$ 上. 于是可取 $V = V_1 \cap V_2 \cap \cdots \cap V_k - f(M - U_1 - \cdots - U_k)$.]

## 1.3 代数基本定理

应用这些概念, 我们证明代数基本定理: *每一个非常值的复多项式 $P(z)$ 必定有一个零点.*

为了证明该定理, 首先要从复数平面过渡到紧致流形. 考虑单位球面 $S^2 \subset \mathbf{R}^3$ 以及从 $S^2$ 的 "北极" $(0,0,1)$ 出发的*球极平面投影* (stereographic projection)

$$h_+ : S^2 - \{(0,0,1)\} \longrightarrow \mathbf{R}^2 \times 0 \subset \mathbf{R}^3.$$

(见图 3.) 我们将把 $\mathbf{R}^2 \times 0$ 与复数平面视为一体. 从 $\mathbf{R}^2 \times 0$ 到自身的

多项式映射 $P$ 对应于从 $S^2$ 到自身的一个映射 $f$, 其中

$$f(x)=\begin{cases} h_+^{-1}Ph_+(x), & 当\ x\neq(0,0,1);\\ (0,0,1), & 当\ x=(0,0,1).\end{cases}$$

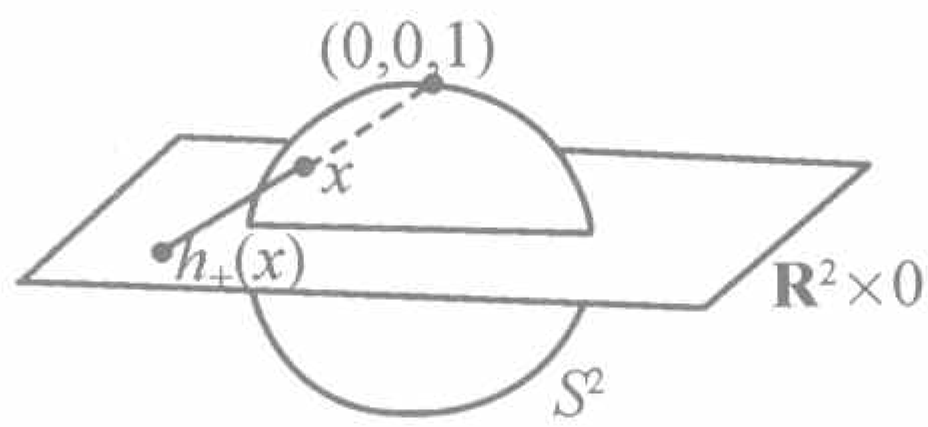


图 3 球极平面投影

众所周知, 得到的映射 $f$ 是光滑的, 即使在北极的一个邻域中也是如此. 为了看清这一点, 引进从南极 $(0,0,-1)$ 出发的球极平面投影 $(h_-)$, 且令

$$Q(z)=h_-fh_-^{-1}(z).$$

注意: 根据初等几何, 有

$$h_+h_-^{-1}(z)=z/|z|^2=1/\bar{z}.$$

现若 $P(z)=a_0z^n+a_1z^{n-1}+\cdots+a_n$, 其中 $a_0\neq 0$, 则通过一个简短的计算, 可得到

$$Q(z)=z^n/(\bar{a}_0+\bar{a}_1z+\cdots+\bar{a}_nz^n).$$

于是 $Q$ 在 0 的一个邻域中是光滑的, 这就推出 $f=h_-^{-1}Qh_-$ 在 $(0,0,1)$ 的一个邻域中是光滑的,

其次, 注意: $f$ 只有有限个临界点, 因为 $P$ 仅在微商多项式 $P'(z)=\Sigma a_{n-j}jz^{j-1}$ 的零点处才不是局部微分同胚, 且由于 $P'$ 不恒等于零, 它只有有限多个零点. $f$ 的正则值的集合是除去了有限个点的球, 因而是连通的. 所以, 在这个集合上局部常值函数 $\#f^{-1}(y)$ 自然应当是常值的. 因为 $\#f^{-1}(y)$ 不能处处为零, 所以我们断定它无处为 0. 于是 $f$ 是一个满射, 从而多项式 $P$ 必有零点.

# 第 2 章　Sard 定理和 Brown 定理

一般地说, 如果希望光滑映射的临界值的集合是有限的, 那就太过分了. 但是, 在下述定理所说明的意义下, 这个集合是"微小的". 这个定理是由 A. Sard 于 1942 年在 A. P. Morse 较早工作的基础上证明的. (参见 [30]、[24].)

**定理**　设 $f: U \to \mathbf{R}^n$ 为定义在开集 $U \subset \mathbf{R}^m$ 上的一个光滑映射, 并令

$$C = \{x \in U \mid \mathrm{d}f_x \text{ 的秩 } < n\},$$

则象 $f(C) \subset \mathbf{R}^n$ 的 Lebesgue 测度为零[①].

因为零测集不能包含任何非空开集, 因而余集 $\mathbf{R}^n - f(C)$ 必定在 $\mathbf{R}^n$ 中处处稠密[②].

定理的证明将在第 3 章中给出. 这个证明的关键在于 $f$ 应当有多重偏导数. (参见 Whitney[38].)

我们主要对 $m \geqslant n$ 的情形感兴趣. 若 $m < n$, 则显然 $C = U$; 因此本定理只是简单地谈到 $f(U)$ 的测度为零.

更一般些, 考虑从 $m$ 维流形到 $n$ 维流形的一个光滑映射 $f: M \to N$. 设 $C$ 为所有使得

$$\mathrm{d}f_x : TM_x \to TN_{f(x)}$$

的秩小于 $n$ 的 (即 $\mathrm{d}f_x$ 不是满的) 点 $x \in M$ 的集合. 则 $C$ 称为**临界点集**, $f(C)$ 称为**临界集**, 而余集 $N - f(C)$ 则称为 $f$ 的**正则值集**. (当 $m = n$ 时, 这与我们先前的定义相合.) 由于 $M$ 能被邻域的可数集族所覆盖, 这些邻域的每一个同胚于 $\mathbf{R}^m$ 的一个开子集, 故有:

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① 换言之, 对于任意给定的 $\epsilon > 0$, 总有一个 $n$ 维总体积小于 $\epsilon$ 的 $\mathbf{R}^n$ 中的方体序列覆盖 $f(C)$.

② Arthur B. Brown 于 1935 年证明. 该结论又由 Dubovickiĭ (于 1953 年) 以及 Thom (于 1954) 年重新发现. (参见 [5]、[8]、[36].)

**推论 (A. B. Brown)** 任何一个光滑映射 $f: M \to N$ 的正则值集在 $N$ 中是处处稠密的.

为了阐发这一推论, 我们需要下列引理:

**引理 1** 若 $f: M \to N$ 为一个从 $m$ 维流形到 $n$ 维流形的光滑映射 $(m \geqslant n)$, 并且 $y \in N$ 为一个正则值, 则集合 $f^{-1}(y) \subset M$ 为一个 $m-n$ 维光滑流形.

**证明** 令 $x \in f^{-1}(y)$. 因为 $y$ 是一个正则值, 导射 $\mathrm{d}f_x$ 必将 $TM_x$ 映射到 $TN_y$ 上. $\mathrm{d}f_x$ 的零空间 $\mathfrak{R} \subset TM_x$ 是一个 $m-n$ 维向量空间.

若 $M \subset \mathbf{R}^k$, 选取一个在子空间 $\mathfrak{R} \subset TM_x \subset \mathbf{R}^k$ 上非退化的线性映射 $L: \mathbf{R}^k \to \mathbf{R}^{m-n}$. 现在用 $F(\xi) = (f(\xi), L(\xi))$ 定义

$$F: M \longrightarrow N \times \mathbf{R}^{m-n}.$$

导射 $\mathrm{d}F_x$ 显然由公式

$$\mathrm{d}F_x(v) = (\mathrm{d}f_x(v), L(v))$$

给出. 于是 $\mathrm{d}F_x$ 是非退化的. 因此 $F$ 将 $x$ 的某一邻域 $U$ 微分同胚地映射到 $(y, L(x))$ 的某一个邻域 $V$ 上. 注意 $f^{-1}(y)$ 在映射 $F$ 下对应着超平面 $y \times \mathbf{R}^{m-n}$. 事实上, $F$ 将 $f^{-1}(y) \cap U$ 微分同胚地映射到 $(y \times \mathbf{R}^{m-n}) \cap V$ 上. 这证明了 $f^{-1}(y)$ 是一个 $m-n$ 维光滑流形.

作为例子, 我们可以给出单位球面 $S^{m-1}$ 是一个光滑流形的简单证明. 考虑函数 $f: \mathbf{R}^m \to \mathbf{R}$, 其定义为

$$f(x) = x_1^2 + x_2^2 + \cdots + x_m^2.$$

任意 $y \neq 0$ 都是正则值, 因而光滑流形 $f^{-1}(1)$ 是一个单位球面.

若 $M'$ 是包含在 $M$ 中的一个流形, 注意到对于 $x \in M'$, $TM'_x$ 是 $TM_x$ 的子空间. 于是 $TM'_x$ 在 $TM_x$ 中的正交补是一个 $m-m'$ 维的向量空间, 称为 $M'$ 在 $M$ 中点 $x$ 处的**法向量空间**(the space of normal vectors).

特别地, 对于 $f: M \to N$ 的某一个正则值 $y$, 令 $M' = f^{-1}(y)$.

**引理 2** $\mathrm{d}f_x: TM_x \to TN_y$ 的零空间恰好等于子流形[①] $M' = f^{-1}(y)$ 的切空间 $TM'_x \subset TM_x$. 因此, $\mathrm{d}f_x$ 将 $TM'_x$ 的正交补同构地映射到 $TN_y$ 上.

① 若流形 $M' \subset$ 流形 $M$, 则称 $M'$ 是 $M$ 的子流形. —— 译者注

**证明** 由图

$$\begin{tikzcd} M' \arrow[r, "i"] \arrow[d] & M \arrow[d, "f"] \\ y \arrow[r] & N \end{tikzcd}$$

可以知道, $\mathrm{d}f_x$ 将子空间 $TM'_x \subset TM_x$ 映射到零. 通过计算维数可见, $\mathrm{d}f_x$ 将 $M'$ 的法向量空间同构地映射到 $TN_y$ 上.

## 2.1 有边流形

可以修改上述引理, 以适用于定义在光滑的“有边流形”上的映射. 首先考虑闭的半空间

$$H^m = \{(x_1, \cdots, x_m) \in \mathbf{R}^m \mid x_m \geqslant 0\},$$

边 (boundary) $\partial H^m$ 定义为超平面 $\mathbf{R}^{m-1} \times 0 \subset \mathbf{R}^m$.

**定义** 设 $X \subset \mathbf{R}^k$. 如果每一个 $x \in X$ 有一邻域 $U \cap X$ 微分同胚于 $H^m$ 的一个开子集 $V \cap H^m$, 则 $X$ 称为*有边的光滑 $m$ 维流形* (smooth $m$-manifold with boundary). *边* $\partial X$ 是在这种微分同胚下 $X$ 中对应于 $\partial H^m$ 的点的所有那些点构成的集合.

不难证明 $\partial X$ 是一个有确切定义的 $m-1$ 维光滑流形, 其*内部* (interior) $X - \partial X$ 是一个 $m$ 维光滑流形.

如在第 1 章中那样定义切空间 $TX_x$, 即使 $x$ 是边点, $TX_x$ 也是整个的 $m$ 维向量空间.

在此有一个给出有边流形例子的方法. 设 $M$ 为一个无边流形, $g: M \to R$ 以 0 为它的一个正则值.

**引理 3** *$M$ 中使得 $g(x) \geqslant 0$ 的点 $x$ 构成的集合是一个光滑的有边流形, 其边等于 $g^{-1}(0)$.*

此引理的证明与引理 1 的证明完全类似.

**例** 由所有满足条件

$$1 - \Sigma x_i^2 \geqslant 0$$

的点 $x \in \mathbf{R}^m$ 组成的单位圆盘 (unit disk) $D^m$ 是一个光滑的有边流形, 其边等于 $S^{m-1}$.

现在考虑从有边 $m$ 维流形到 $n$ 维流形的一个光滑映射 $f : X \to N$, 其中 $m > n$.

**引理 4** *若对 $f$ 而言以及对限制映射 $f \mid \partial X$ 而言, $y \in N$ 都是正则值, 则 $f^{-1}(y) \subset X$ 是一个光滑的有边 $(m-n)$ 维流形. 并且边 $\partial(f^{-1}(y))$ 正好等于 $f^{-1}(y)$ 与 $\partial X$ 的交.*

**证明** 因为要证明的是一个局部性质, 所以只要考虑以下特殊情形: $y \in \mathbf{R}^n$ 是映射 $f : H^m \to \mathbf{R}^n$ 的一个正则值. 令 $\overline{x} \in f^{-1}(y)$. 若 $\overline{x}$ 是一个内点, 则如前可见, $f^{-1}(y)$ 在 $\overline{x}$ 的邻域中是一个光滑流形.

假设 $\overline{x}$ 是一个边点. 选取一个光滑映射 $g : U \to \mathbf{R}^n$, $g$ 定义于 $\overline{x}$ 在 $\mathbf{R}^m$ 中的整个邻域 $U$ 上, 并且在 $U \cap H^m$ 上与 $f$ 一致. 如果必要的话, 用小一些的邻域替换 $U$, 我们可以假定 $g$ 没有临界点. 因此 $g^{-1}(y)$ 是一个 $m-n$ 维光滑流形.

设 $\pi : g^{-1}(y) \to R$ 表示坐标射影,

$$\pi(x_1, \cdots, x_m) = x_m.$$

可以断定 $\pi$ 以 0 为一个正则值. 因为 $g^{-1}(y)$ 在点 $x \in \pi^{-1}(0)$ 处的切空间等于

$$\mathrm{d}g_x = \mathrm{d}f_x : \mathbf{R}^m \longrightarrow \mathbf{R}^n$$

的零空间, 而 $f \mid \partial H^m$ 在点 $x$ 处是正则的这一假定保证了这个零空间不能完全包含在 $\mathbf{R}^{m-1} \times 0$ 中.

因此根据引理 3, 由所有满足 $\pi(x) \geqslant 0$ 的点 $x \in g^{-1}(y)$ 组成的集合 $g^{-1}(y) \cap H^m = f^{-1}(y) \cap U$ 是光滑的有边流形, 其边等于 $\pi^{-1}(0)$. 这就完成了证明.

## 2.2 Brouwer 不动点定理

现在应用上述结果来证明一个关键性引理, 利用它可以推导出经典的 Brouwer 不动点定理. 设 $X$ 为一个紧致的有边流形.

**引理 5** *不存在光滑映射 $f : X \to \partial X$ 保持 $\partial X$ 点式不动.*

**证明 (遵循 M. Hirsch 的证明方式)** 假设存在着这样的一个映射 $f$. 设 $y \in \partial X$ 为 $f$ 的一个正则值. 因为 $y$ 肯定也是恒同映射 $f \mid \partial X$ 的正则值, 所以 $f^{-1}(y)$ 是光滑的有边 1 维流形, 其边由独点

$$f^{-1}(y) \cap \partial X = \{y\}$$

组成. 但 $f^{-1}(y)$ 也是紧致的, 而仅有的紧致 1 维流形都是有限个圆周和闭线段的不相交的并①, 所以 $f^{-1}(y)$ 必定由偶数个点组成. 这一矛盾便证实了引理.

特别地, 单位圆盘

$$D^n = \{x \in \mathbf{R}^n \mid x_1^2 + \cdots + x_n^2 \leqslant 1\}$$

是以单位球 $S^{n-1}$ 为边的紧致流形. 因此作为特殊情形, 我们证明了 $S^{n-1}$ 的恒同映射不能扩充为光滑映射 $D^n \to S^{n-1}$.

**引理 6** 任何光滑映射 $g: D^n \to D^n$ 都有一个不动点 (即一个点 $x \in D^n$ 使得 $g(x) = x$).

**证明** 假设 $g$ 没有不动点. 对于 $x \in D^n$, 令 $f(x) \in S^{n-1}$ 为在通过 $x$ 与 $g(x)$ 的直线上的距离 $x$ 比距离 $g(x)$ 更近的那个点 (见图 4), 则 $f: D^n \to S^{n-1}$ 为一个光滑映射, 且对于 $x \in S^{n-1}$, $f(x) = x$. 据引理 5 知, 这是不可能的. (为了知道 $f$ 是光滑的, 我们给出下列解析表达式: $f(x) = x + tu$, 其中

$$u = \frac{x - g(x)}{\|x - g(x)\|}, \quad t = -x \cdot u + \sqrt{1 - x \cdot x + (x \cdot u)^2},$$

在平方根符号下的表达式严格为正. 此处以及以后的 $\|x\|$ 都表示欧氏长度 $\sqrt{x_1^2 + \cdots + x_n^2}$.)

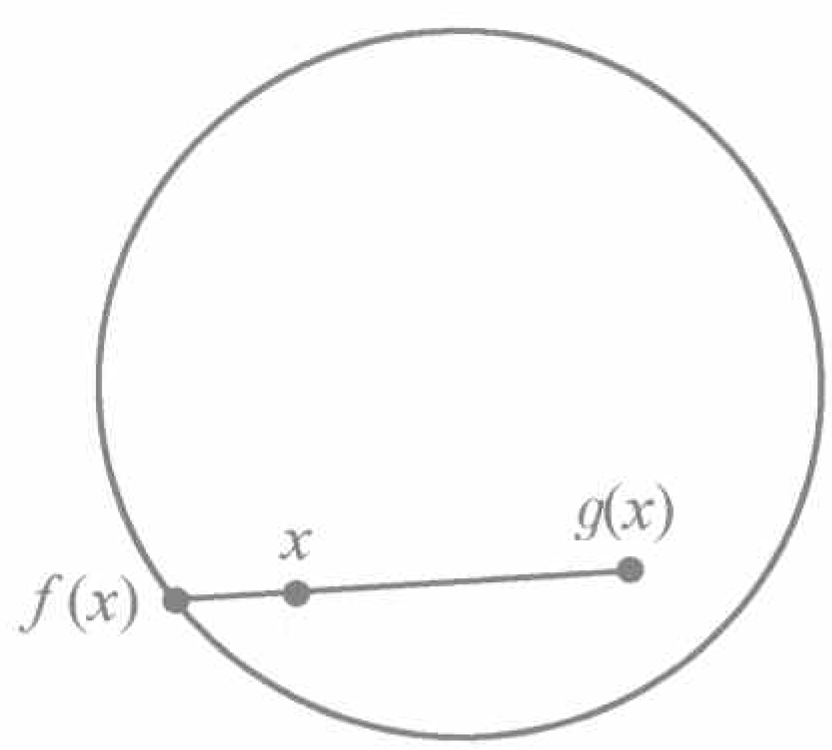


图 4

① 附录中给出了一个证明.

**Brouwer 不动点定理** 任何连续映射 $G: D^n \to D^n$ 都有一个不动点.

**证明** 我们用光滑映射来逼近 $G$, 通过引理来证明该定理. 给定 $\epsilon > 0$, 根据 Weierstrass 逼近定理①, 存在多项式函数 $P_1: \mathbf{R}^n \to \mathbf{R}^n$ 满足条件: 对于 $x \in D^n$, $\| P_1(x) - G(x) \| < \epsilon$. 然而, $P_1$ 可能把 $D^n$ 的点变为 $D^n$ 外面的点. 为了修正这一点, 设

$$P(x) = P_1(x)/(1+\epsilon).$$

则显然 $P$ 将 $D^n$ 映射到 $D^n$ 中, 并且对于 $x \in D^n$, $\| P(x) - G(x) \| < 2\epsilon$.

假设对于所有 $x \in D^n$, $G(x) \neq x$. 则连续函数 $\| G(x) - x \|$ 必定在 $D^n$ 上取到一个最小值 $\mu > 0$. 如上选取 $P: D^n \to D^n$ 使得对于所有 $x$, $\| P(x) - G(x) \| < \mu$. 显然有 $P(x) \neq x$. 于是 $P$ 是一个从 $D^n$ 到自身的且没有不动点的光滑映射. 这与引理 6 矛盾. 于是证明完成.

这里用到的以下方法常常能够用于更一般的情形: 为了证明关于连续映射的一个命题, 首先对于光滑映射建立这一结果, 然后用一个逼近定理过渡到连续情形. (参见第 8 章, 问题 4.)

① 例子可参见 Dieudonne [7. p.133].

# 第 3 章 Sard 定理的证明①

首先重新陈述 Sard 定理如下:

**Sard 定理** 设 $f: U \to \mathbf{R}^p$ 为一个光滑映射, 其中 $U$ 是 $\mathbf{R}^n$ 中的一个开集. 令 $C$ 为临界点集, 即所有使得

$$\mathrm{d}f_x \text{ 的秩 } < p$$

的点 $x \in U$ 的集合. 则 $f(C) \subset \mathbf{R}^p$ 的测度为零.

**注记** $n \leqslant p$ 的情形是比较容易的. (参见 de Rham [29, p.10].) 不过, 我们要给一个统一的证明.

**证明** 对 $n$ 作归纳. 注意定理的陈述对于 $n \geqslant 0$, $p \geqslant 1$ 都有意义. (据定义, $\mathbf{R}^0$ 由独点组成.) 作为归纳的开始, 当 $n = 0$ 时定理肯定为真.

令 $C_1 \subset C$ 表示所有使得一阶导射 $\mathrm{d}f_x$ 为零的点 $x \in U$ 组成的集合. 更一般些, 令 $C_i$ 表示使得 $f$ 的所有阶数 $\leqslant i$ 的偏导为零的那些点 $x$ 的集合. 于是得到一个闭集递减序列

$$C \supset C_1 \supset C_2 \supset C_3 \supset \cdots.$$

证明分如下 3 个步骤:

**第 1 步** 象 $f(C - C_1)$ 测度为零.

**第 2 步** 当 $i \geqslant 1$ 时, 象 $f(C_i - C_{i+1})$ 测度为零.

**第 3 步** 当 $k$ 足够大时, 象 $f(C_k)$ 测度为零.

(**注记** 若 $f$ 是实解析的, 则所有 $C_i$ 的交是空的, 除非 $f$ 在 $U$ 的某一分支上为常值. 因此在这种情形下只要给出第 1 步和第 2 步.)

**第 1 步的证明** 这一步也许是最难的一步, 我们可以假定 $p \geqslant 2$,

---

①·这里的证明是根据 Pontryagin[28] 中的证明给出的, 只是在细节上稍微简单一点, 因为这里假定 $f$ 是无限次可微的.

因为当 $p=1$ 时 $C=C_1$. 我们需要众所周知的 Fubini 定理[①]: 一个可测集

$$A \subset \mathbf{R}^p = \mathbf{R}^1 \times \mathbf{R}^{p-1}$$

如果与每一个超平面 (常数) $\times \mathbf{R}^{p-1}$ 交于 $p-1$ 维零测度集, 则 $A$ 的测度必定为零.

对于每一个 $\overline{x} \in C - C_1$, 我们找一个开邻域 $V \subset \mathbf{R}^n$, 使得 $f(V \cap C)$ 测度为零. 因为 $C - C_1$ 被可数多个这种邻域所覆盖, 这就证明了 $f(C - C_1)$ 测度为零.

由于 $\overline{x} \notin C_1$, 存在某一个偏导数, 设为 $\partial f_1/\partial x_1$, 在点 $\overline{x}$ 处不为零. 考虑由 $h(x) = (f_1(x), x_2, \cdots, x_n)$ 定义的映射 $h: U \to \mathbf{R}^n$, 因为 $\mathrm{d}h_x$ 是非退化的, $h$ 将 $\overline{x}$ 的某一邻域 $V$ 微分同胚地映射到一个开集 $V'$ 上. 于是复合映射 $g = f \circ h^{-1}$ 将 $V'$ 映射到 $\mathbf{R}^p$ 中. 注意 $g$ 的临界点集 $C'$ 正好是 $h(V \cap C)$. 因此 $g$ 的临界值的集合 $g(C')$ 等于 $f(V \cap C)$.

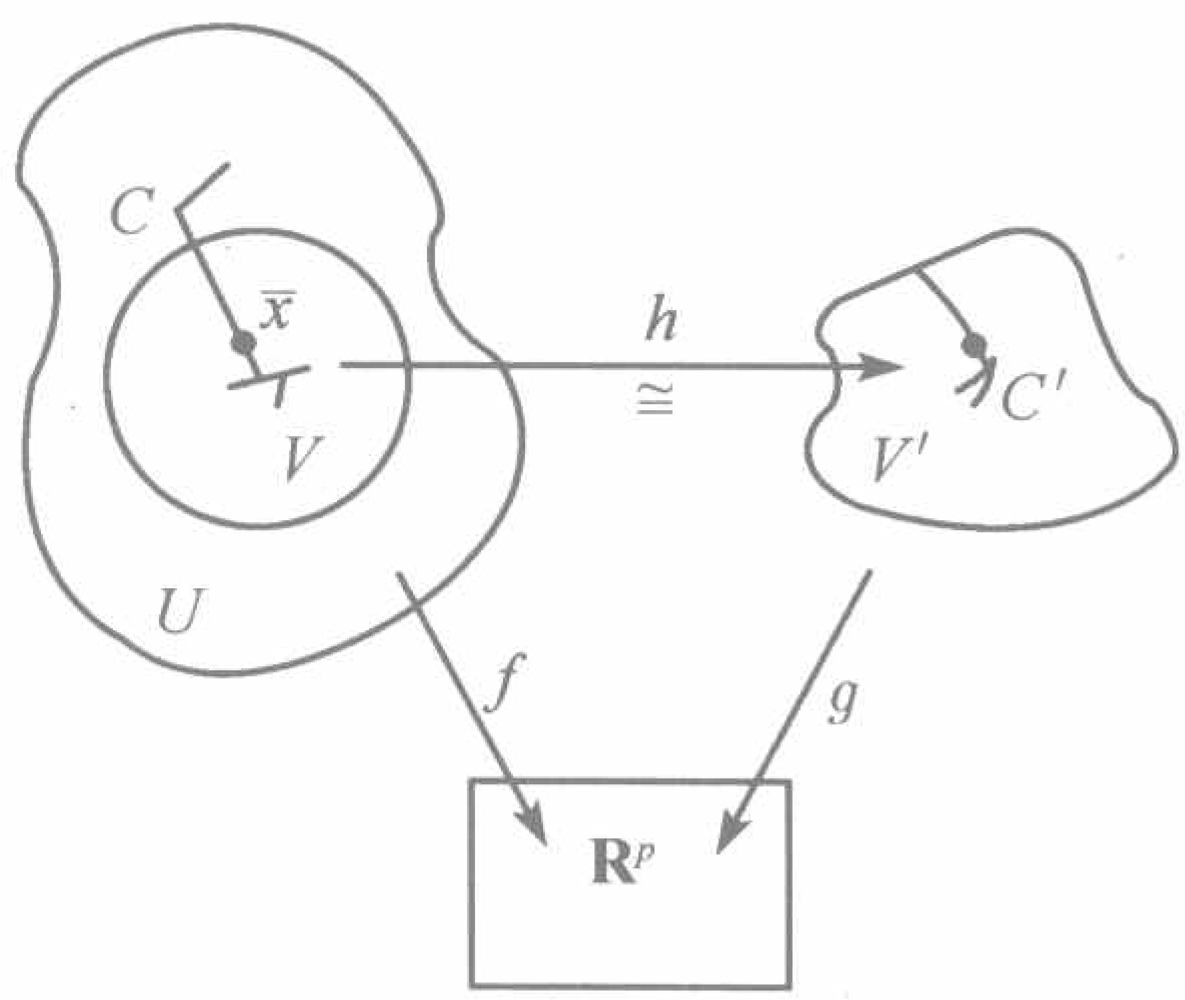


图 5 映射 $g$ 的构造

对于每一个 $(t, x_2, \cdots, x_n) \in V'$, 注意 $g(t, x_2, \cdots, x_n)$ 属于超平面 $t \times \mathbf{R}^{p-1} \subset \mathbf{R}^p$, 于是 $g$ 将超平面变到超平面中. 令

$$g^t : (t \times \mathbf{R}^{n-1}) \cap V' \longrightarrow t \times \mathbf{R}^{p-1}$$

表示 $g$ 的限制. 注意 $t \times \mathbf{R}^{n-1}$ 的点是 $g^t$ 的临界点当且仅当它是 $g$ 的

① 一个简单的证明 (以及 Sard 定理的另外一个证明) 见 Sternberg [35, pp.51-52]. Sternberg 假定 $A$ 是紧致的, 不过一般情形可以很容易地从这种特殊情形推出来.

临界点, 因为 $g$ 的一阶偏导矩阵形如

$$(\partial g_i/\partial x_j) = \begin{bmatrix} 1 & 0 \\ * & (\partial g_i^t/\partial x_j) \end{bmatrix}.$$

根据归纳假设, $g^t$ 的临界值集在 $t \times \mathbf{R}^{p-1}$ 中测度为零, 因此 $g$ 的临界值集与每一个超平面 $t \times \mathbf{R}^{p-1}$ 交于一个零测度集. 集合 $g(C')$ 是可测的, 因为它能表为紧致子集的可数并. 因此, 根据 Fubini 定理, 集合

$$g(C') = f(V \cap C)$$

测度为零, 第 1 步证完.

**第 2 步的证明**　对于每一个 $\overline{x} \in C_k - C_{k+1}$, 存在某一个 $k+1$ 阶偏导数 $\partial^{k+1} f_r/\partial x_{s_1} \cdots \partial x_{s_{k+1}}$ 不为零. 于是函数

$$w(x) = \partial^k f_r/\partial x_{s_2} \cdots \partial x_{s_{k+1}}$$

在点 $\overline{x}$ 处为零, 但 $\partial w/\partial x_{s_1}$ 在点 $\overline{x}$ 处不为零. 为确定起见, 设 $s_1 = 1$ 则由

$$h(x) = (w(x), x_2, \cdots, x_n)$$

定义的映射 $h: U \to \mathbf{R}^n$ 将 $\bar{x}$ 的某一邻域 $V$ 微分同胚地变到某一个开集 $V'$ 上. 注意 $h$ 将 $C_k \cap V$ 变到超平面 $0 \times \mathbf{R}^{n-1}$ 中. 我们再考虑

$$g = f \circ h^{-1} : V' \longrightarrow \mathbf{R}^p.$$

令

$$\overline{g} : (0 \times \mathbf{R}^{n-1}) \cap V' \longrightarrow \mathbf{R}^p$$

表示 $g$ 的限制. 由归纳法知, $\overline{g}$ 的临界值集在 $\mathbf{R}^p$ 中测度为零. 但 $h(C_k \cap V)$ 中的每一个点肯定是 $\overline{g}$ 的临界点 (因为所有阶数 $\leqslant k$ 的偏导数为零). 因此

$$\overline{g}h(C_k \cap V) = f(C_k \cap V)$$

测度为零. 由于 $C_k - C_{k+1}$ 是被可数多个这种集合 $V$ 所覆盖, 从而推得 $f(C_k - C_{k+1})$ 测度为零.

**第 3 步的证明** 令 $I^n \subset U$ 为边长等于 $\delta$ 的一个立方体. 若 $k$ 充分大 (确切地说 $k > n/p - 1$), 可以证明 $f(C_k \cap I^n)$ 测度为零. 因为 $C_k$ 能被可数多个这种立方体所覆盖, 这就证明了 $f(C_k)$ 测度为零.

根据 Taylor 定理、$I^n$ 的紧致性以及 $C_k$ 的定义可知, 当 $x \in C_k \cap I^n$, $x + h \in I^n$ 时,

$$f(x+h) = f(x) + R(x,h),$$

其中

$$\| R(x,h) \| \leqslant c \, \| h \|^{k+1}, \tag{1}$$

这里 $c$ 为仅依赖于 $f$ 和 $I^n$ 的常数. 现在重分 $I^n$ 为边长等于 $\delta/r$ 的 $r^n$ 个立方体. 令 $I_1$ 为重分中的一个立方体, 它包含 $C_k$ 的一个点 $x$. 则 $I_1$ 的任何一点能写作 $x + h$, 其中

$$\| h \| \leqslant \sqrt{n}(\delta/r). \tag{2}$$

从 (1) 推出 $f(I_1)$ 在一个边长为 $a/r^{k+1}$ 且以 $f(x)$ 为中心的立方体中, 其中 $a = 2c(\sqrt{n}\delta)^{k+1}$ 是常数. 因此 $f(C_k \cap I^n)$ 包含在最多 $r^n$ 个立方体的并中, 这些立方体总体积为

$$V \leqslant r^n (a/r^{k+1})^p = a^p r^{n-(k+1)p}.$$

若 $k + 1 > n/p$, 则显然当 $r \to \infty$ 时, $V$ 趋近于 0, 所以 $f(C_k \cap I^n)$ 必定测度为零. Sard 定理证毕.

# 第 4 章　映射的模 2 度

考虑一个光滑映射 $f: S^n \to S^n$. 若 $y$ 为一个正则值, $\# f^{-1}(y)$ 表示方程 $f(x) = y$ 的解 $x$ 的个数. 可以证明 $\# f^{-1}(y)$ 的模 2 同余类不依赖于正则值 $y$ 的选取. 这一同余类称为 $f$ 的模 2 度 (mod 2 degree). 更一般地, 上述定义对于任何光滑映射

$$f: M \longrightarrow N$$

都有效, 其中 $M$ 是紧致的无边流形, $N$ 是连通的, 并且两个流形有相同的维数 (此外, 也可以假定 $N$ 是紧致无边的, 因为否则, 模 2 度必为零). 为了证明上述论断, 下面引进两个新概念.

## 光滑同伦和光滑同痕

给定 $X \subset \mathbf{R}^k$, 令 $X \times [0,1]$ 表示由所有满足 $x \in X$ 以及 $0 \leqslant t \leqslant 1$ 的 $(x, t)$ 组成的 $\mathbf{R}^{k+1}$ 的子集①. 两个映射

$$f, g: X \longrightarrow Y$$

称为是*光滑同伦的* (smoothly homotopic) (简记作 $f \sim g$), 如果存在一个光滑映射 $F: X \times [0,1] \to Y$ 满足条件: 对于所有 $x \in X$,

$$F(x, 0) = f(x), \qquad F(x, 1) = g(x).$$

这一映射 $F$ 称为 $f$ 与 $g$ 之间的一个*光滑同伦* (smooth homotopy).

注意: 光滑同伦的关系是一个等价关系. 为了证明它是传递的, 我们要用到: 存在一个光滑函数 $\varphi: [0,1] \to [0,1]$, 满足条件

$$\varphi(t) = \begin{cases} 0, & \text{当 } 0 \leqslant t \leqslant \frac{1}{3}; \\ 1, & \text{当 } \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

---

① 若 $M$ 为一个光滑的无边流形, 则 $M \times [0,1]$ 是以 $M$ 的两个“副本”为边的光滑流形. $M$ 的边点将会生出 $M \times [0,1]$ 的“角”点.

(例如令 $\varphi(t) = \lambda\left(t-\frac{1}{3}\right) / \left[\lambda\left(t-\frac{1}{3}\right)+\lambda\left(\frac{2}{3}-t\right)\right]$, 其中当 $\tau \leqslant 0$ 时, $\lambda(\tau)=0$; 而当 $\tau>0$ 时, $\lambda(\tau)=\exp(-\tau^{-1})$.) 给定 $f$ 与 $g$ 之间的一个光滑同伦 $F$, 公式 $G(x,t)=F(x,\varphi(t))$ 确定了一个光滑同伦 $G$, 满足条件:

$$G(x,t)=\begin{cases} f(x), & \text{当 } 0 \leqslant t \leqslant \frac{1}{3}; \\ g(x), & \text{当 } \frac{2}{3} \leqslant t \leqslant 1. \end{cases}$$

现在如果 $f \sim g$ 和 $g \sim h$, 则利用上面的作法容易证明 $f \sim h$.

若 $f$ 与 $g$ 都是从 $X$ 到 $Y$ 的微分同胚, 我们也能定义 $f$ 与 $g$ 之间的"光滑同痕"的概念, 这也是一个等价关系.

**定义** 称微分同胚 $f$ 是光滑同痕于 $g$ 的, 如果存在一个从 $f$ 到 $g$ 的光滑同伦 $F: X\times[0,1]\to Y$, 使得对于每一个 $t\in[0,1]$, 对应

$$x \to F(x,t)$$

将 $X$ 微分同胚地映射到 $Y$ 上.

下面要证明一个映射的模 2 度仅仅依赖于它的光滑同伦类. 为此, 先证明两个引理:

**同伦引理** 令 $f,g: M\to N$ 为光滑同伦的两个映射, 其中 $M$, $N$ 是两个具有相同维数的流形, 并且 $M$ 是紧致无边的. 若 $y\in N$ 对于 $f$ 与 $g$ 两者都是正则值, 则

$$\# f^{-1}(y) \equiv \# g^{-1}(y) \pmod 2.$$

**证明** 令 $F: M\times[0,1]\to N$ 为 $f$ 与 $g$ 之间的一个光滑同伦. 先设 $y$ 也是 $F$ 的正则值, 则 $F^{-1}(y)$ 是一个紧致 1 维流形, 其边等于

$$F^{-1}(y)\cap(M\times 0\cup M\times 1)=f^{-1}(y)\times 0\cup g^{-1}(y)\times 1,$$

于是 $F^{-1}(y)$ 的边点的总数等于

$$\# f^{-1}(y)+\# g^{-1}(y).$$

但在第 2 章中说过, 紧致的 1 维流形总有偶数个边点. 所以 $\# f^{-1}(y)+\# g^{-1}(y)$ 是偶数, 因此

$$\# f^{-1}(y) \equiv \# g^{-1}(y) \pmod 2.$$

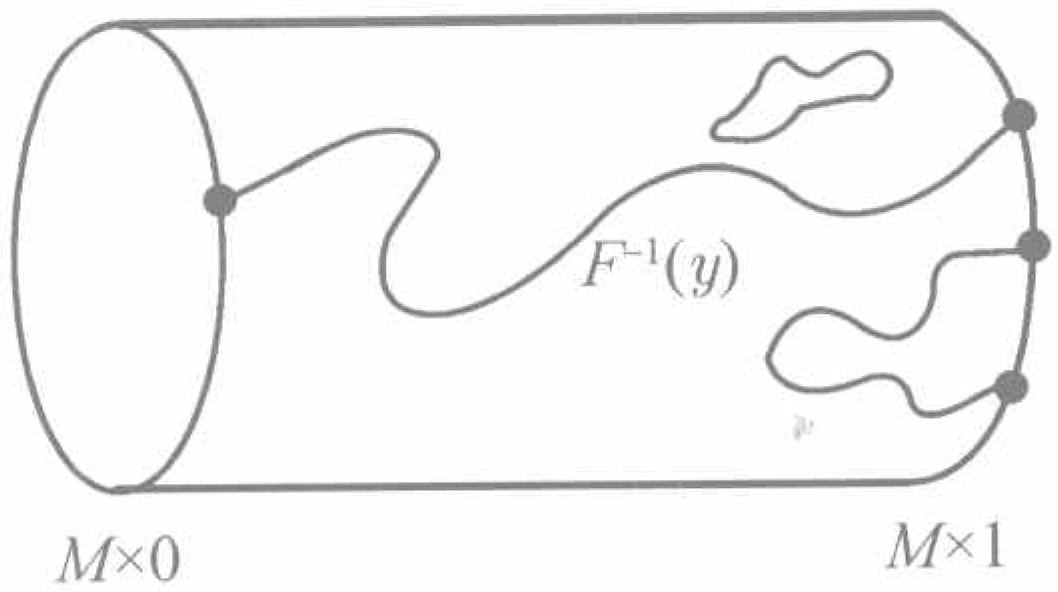


图 6　在左边的边点数与在右边的边点数模 2 同余

现在假设 $y$ 不是 $F$ 的正则值. 在第 1 章中说过, $\#f^{-1}(y')$ 以及 $\#g^{-1}(y')$ 都是 $y'$ 的局部常值函数 (只要避开临界值), 于是存在一个由 $f$ 的正则值组成的 $y$ 的邻域 $V_1\subset N$, 使得对于所有的 $y'\in V_1$ 都有

$$\#f^{-1}(y')=\#f^{-1}(y).$$

同时存在一个类似的邻域 $V_2\subset N$, 使得对于所有的 $y'\in V_2$ 都有

$$\#g^{-1}(y')=\#g^{-1}(y).$$

在 $V_1\cap V_2$ 中选取一个 $F$ 的正则值 $z$, 则

$$\#f^{-1}(y)=\#f^{-1}(z)\equiv\#g^{-1}(z)=\#g^{-1}(y).$$

同伦引理证毕.

**齐性引理**　*令 $y$ 与 $z$ 为光滑的连通流形 $N$ 的任意两个内点. 则存在一个光滑同痕于恒同映射并且将 $y$ 变到 $z$ 的微分同胚 $h: N\to N$.*

(对于特殊情形 $N=S^n$, 证明是容易的: $h$ 可简单地选为将 $y$ 变到 $z$, 且保持所有的正交于, 通过 $y$ 和 $z$ 的平面的向量不动的旋转.)

一般的证明如下. 首先构造一个从 $\mathbf{R}^n$ 到自身的光滑同痕, 满足条件:

(1) 保持单位球体外面的所有点都不动;

(2) 将原点滑到开单位球体中任何一个预定的点上去.

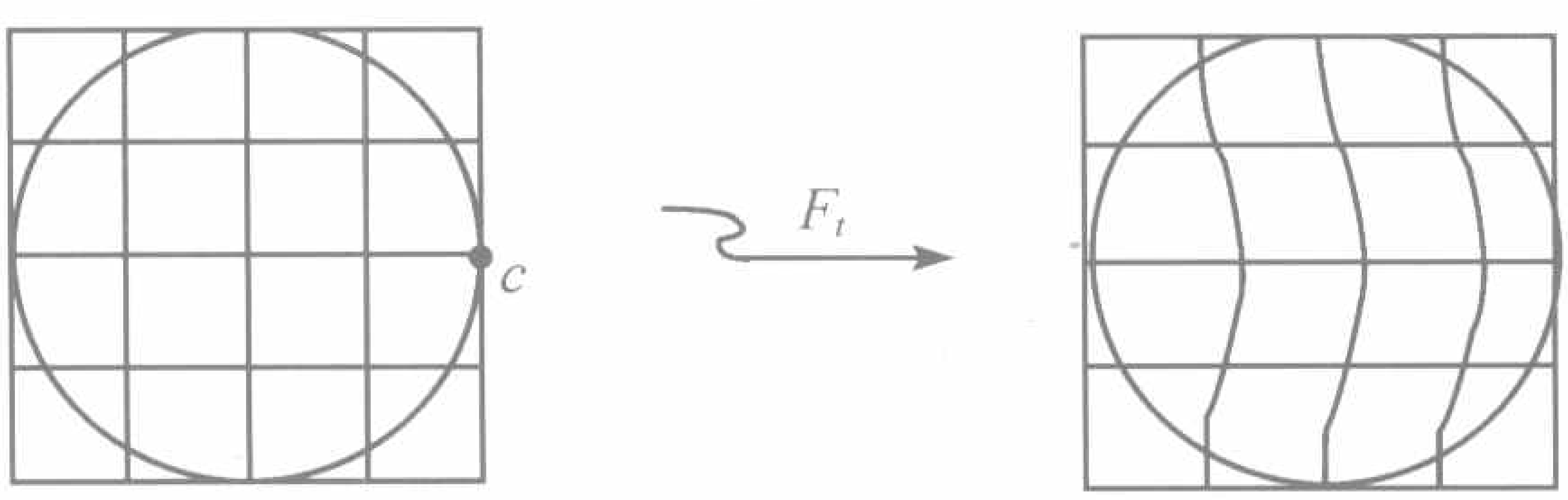


图 7　将单位球体变形

令 $\varphi : \mathbf{R}^n \to \mathbf{R}$ 为一个光滑函数, 满足条件:

$$\text{当 } \|x\| < 1, \quad \varphi(x) > 0; \qquad \text{当 } \|x\| \geqslant 1, \quad \varphi(x) = 0.$$

[例如令 $\varphi(x) = \lambda(1 - \| x \|^2)$, 其中 $\lambda : \mathbf{R} \to \mathbf{R}$ 定义为当 $t \leqslant 0$ 时 $\lambda(t) = 0$; 当 $t > 0$ 时 $\lambda(t) = \exp(-t^{-1})$.] 给定任何一个固定的单位向量 $c \in S^{n-1}$, 考虑微分方程组

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = c_i \varphi(x_1, \cdots, x_n) \quad i = 1, \cdots, n.$$

对于任意的 $\overline{x} \in \mathbf{R}^n$, 该微分方程组有唯一的一个满足初始条件

$$x(0) = \overline{x}$$

的解 $x = x(t)$, 它对于所有实数都有定义①. 使用记号 $x(t) = F_t(\overline{x})$ 来表示这个解. 则显然:

(1) $F_t(\overline{x})$ 对于所有的 $t$ 和 $\overline{x}$ 都有定义, 并且光滑地依赖于 $t$ 和 $\overline{x}$;

(2) $F_0(\overline{x}) = \overline{x}$;

(3) $F_{s+t}(\overline{x}) = F_s \circ F_t(\overline{x})$.

因此每一个 $F_t$ 是从 $\mathbf{R}^n$ 到 $\mathbf{R}^n$ 上的一个微分同胚. 让 $t$ 改变, 我们可以看到在一个保持单位球体外的所有点都不动的同痕下, 每一个 $F_t$ 都光滑同痕于恒同映射. 显然, 适当地选取 $c$ 和 $t$, 微分同胚 $F_t$ 将会把原点变到开单位球体中任何预定的一个点.

现在考虑连通的流形 $N$. 如果存在一个光滑同痕将一个点变为另一个点, 则称 $N$ 的这两个点是“同痕的”. 这明显地是一个等价关系. 若 $y$ 是一个内点, 则它有一个微分同胚于 $\mathbf{R}^n$ 的邻域. 因此, 上述论断表明每一个充分靠近于 $y$ 的点都是“同痕”于 $y$ 的. 换言之, $N$ 的内部的点的每一个“同痕类”都是开集, 进而 $N$ 的内部被剖分为互不相交的开的同痕类, 但 $N$ 的内部是连通的, 因此只有一个同痕类. 引理证毕.

现在就能够证明这一节的主要结果了. 假设 $M$ 紧致且无边, $N$ 为连通的, 并且 $f : M \to N$ 是光滑的.

① 参见 [22, 2.4 节].

**定理**　若 $y$ 与 $z$ 是 $f$ 的一个正则值, 则

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \pmod 2.$$

这一共同的同余类称为 $f$ 的模 2 度, 它仅依赖于 $f$ 的光滑同伦类.

**证明**　给定正则值 $y$ 与 $z$. 令 $h$ 为从 $N$ 到 $N$ 的, 同痕于恒同映射的, 并将 $y$ 变为 $z$ 的一个微分同胚. 则 $z$ 为复合映射 $h \circ f$ 的一个正则值. 因为 $h \circ f$ 同伦于 $f$, 同伦引理断定

$$\#h \circ f^{-1}(z) \equiv \#f^{-1}(z) \pmod 2.$$

但

$$h \circ f^{-1}(z) = f^{-1}h^{-1}(z) = f^{-1}(y),$$

所以

$$\#(h \circ f)^{-1}(z) = \#f^{-1}(y).$$

因此

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \pmod 2.$$

这正是我们所需要的.

这一共同的同余类记为 $\deg_2(f)$. 现设 $f$ 光滑同伦于 $g$. 根据 Sard 定理, 存在一个元素 $y \in N$, 它是 $f$ 与 $g$ 两者的正则值. 同余式

$$\deg_2 f \equiv \#f^{-1}(y) \equiv \#g^{-1}(y) \equiv \deg_2 g \pmod 2$$

表明 $\deg_2(f)$ 是一个光滑同伦不变量. 定理证毕.

**例**　常值映射 $c: M \to M$ 的模 2 度是个偶数①. $M$ 的恒同映射 $I$ 的模 2 度是个奇数. 因此紧致无边流形的恒同映射不同伦于常值映射.

当 $M = S^n$ 时, 这一结果意味着, 没有光滑映射 $f: D^{n+1} \to S^n$ 保持球面上点式不动 (即球面不是圆盘的光滑“收缩核”. 参见第 2 章中的引理 5), 因为这种映射将会引出一个常值映射和恒同映射之间的一个光滑同伦:

$$F: S^n \times [0,1] \longrightarrow S^n, \quad F(x,t) = f(tx).$$

① 这里明显地缺一个条件, 即: $M$ 的维数不为 0. 下面的结论也需要这个条件.

—— 译者注

# 第5章　定 向 流 形

为了把“度”定义为整数 (而不是模 2 整数), 我们必须引进“定向”的概念.

**定义**　有限维实向量空间的一个*定向* (orientation) 乃是有序基的如下的一个等价类: 有序基 $(b_1, \cdots, b_n)$ 与基 $(b'_1, \cdots, b'_n)$ 一样决定*同一定向* (same orientation), 如果 $b'_i = \Sigma a_{ij} b_j$ 满足条件 $\det(a_{ij}) > 0$; 反之, 如果 $\det(a_{ij}) < 0$, 则决定*相反的定向* (opposite orientation). 于是每一个正维数的向量空间恰好有两个定向. 向量空间 $\mathbf{R}^n$ 有一个*标准的定向* (standard orientation), 即基 $(1, 0, \cdots, 0)$, $(0, 1, 0, \cdots, 0)$, $\cdots$, $(0, \cdots, 0, 1)$ 决定的定向.

在零维向量空间的情况, 可以将“定向”定义为符号 $+1$ 或 $-1$.

一个*有向*光滑流形由一个流形 $M$ 与对每一个切空间 $TM_x$ 所选择的一个定向组成. 若 $m \geqslant 1$, 要求它们满足下列条件: 对于 $M$ 的每一个点, 存在一个邻域 $U \subset M$ 以及一个*保持定向的* (orientation preserving), 将 $U$ 映射到 $\mathbf{R}^m$ 或 $H^m$ 的一个开子集上的微分同胚 $h$. 保持定向的意思是: 对于每一个 $x \in U$, 同构 $\mathrm{d}h_x$ 将 $TM_x$ 的指定的定向变为 $\mathbf{R}^m$ 的标准定向.

如果 $M$ 是连通的并且是可定向的, 那么它恰好有两个定向.

如果 $M$ 有边, 我们可以区别在边点处的切空间 $TM_x$ 中的 3 种向量:

(1) 有一些向量与边相切, 形成一个 $m-1$ 维的子空间 $T(\partial M)_x \subset TM_x$;

(2) 有一些“外向的”向量, 形成一个以 $T(\partial M)_x$ 为边的开半空间;

(3) 有一些“内向的”向量, 形成以 $T(\partial M)_x$ 为边的另一个开半空间.

$M$ 的每一个定向决定 $\partial M$ 的一个定向如下: 对于 $x \in \partial M$, 选取 $TM_x$ 的一个正的有向基 $(v_1, v_2, \cdots, v_m)$, 使得 $v_2, \cdots, v_m$ 与边相切

(假定 $m \geqslant 2$), 且 $v_1$ 是“外向的”向量. 那么 $(v_2, \cdots, v_m)$ 便决定了所求的 $\partial M$ 在点 $x$ 处的定向.

如果 $M$ 的维数是 1, 则每一个边点 $x$ 按照正有向向量在 $x$ 点处是内向的或是外向的而指定定向为 $-1$ 或 $+1$ (见图 8).

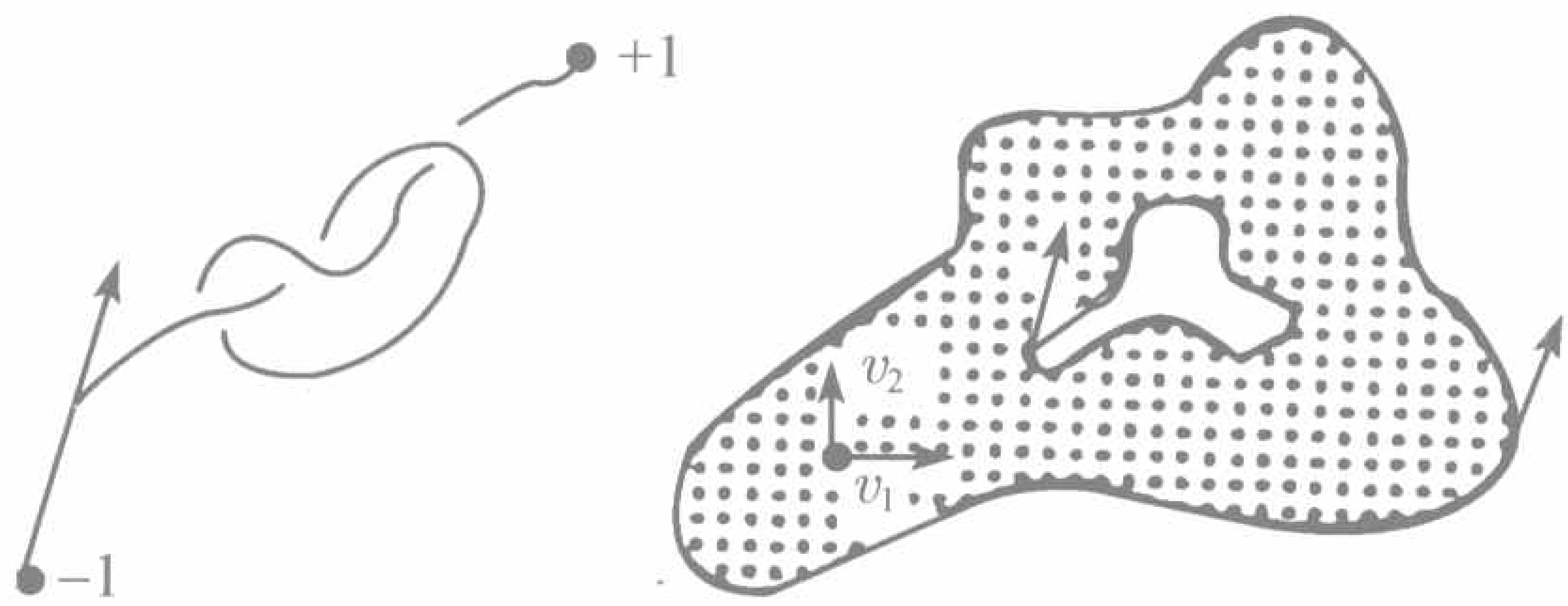


图 8　如何给边定向

例如可以把单位球面 $S^{m-1} \subset \mathbf{R}^m$ 定向为圆盘 $D^m$ 的边.

## Brouwer 度

现设 $M$ 与 $N$ 为两个无边的有向 $n$ 维流形, 且令

$$f : M \longrightarrow N$$

为光滑映射. 若 $M$ 是紧致的, $N$ 是连通的, 则 $f$ 的度定义如下:

设 $x \in M$ 为 $f$ 的一个正则点, 于是 $\mathrm{d}f_x : TM_x \to TN_{f(x)}$ 为有向向量空间之间的一个线性同构. 定义 $\mathrm{d}f_x$ 的符号为 $+1$ 或 $-1$, 按 $\mathrm{d}f_x$ 保持定向或反转定向而定. 对于任何一个正则值 $y \in N$, 定义

$$\deg(f; y) = \sum_{x \in f^{-1}(y)} \operatorname{sign} \mathrm{d}f_x.$$

如在第 1 章中一样, 上式中整数 $\deg(f; y)$ 是 $y$ 的局部常值函数. 它被定义在 $N$ 的一个稠密的开子集上.

**定理 A**　整数 $\deg(f; y)$ 不依赖于正则值 $y$ 的选取.

于是这个整数就称为 $f$ 的度 (degree) (记作 $\deg f$).

**定理 B**　若 $f$ 光滑地同伦于 $g$, 则 $\deg f = \deg g$.

**证明** 本质上与第 4 章中的证明一样. 只是必须一直谨慎地控制定向.

首先考虑下述情形: 设 $M$ 为一个紧致的有向流形 $X$ 的边, 并且 $M$ 是作为 $X$ 的边而予以定向的.

**引理 1** 若 $f: M \to N$ 扩充为一个光滑映射 $F: M \to N$, 则对于每一个正则值 $y$, $\deg(f; y) = 0$.

**证明** 首先设 $y$ 既是 $F$ 的正则值又是 $f = F \mid M$ 的正则值. 紧致的 1 维流形 $F^{-1}(y)$ 为弧和圆周的有限并, 只有弧有边点, 并且在 $M = \partial X$ 上. 令 $A \subset F^{-1}(y)$ 为这些弧中的一个, 且 $\partial A = \{a\} \cup \{b\}$. 下面证明

$$\operatorname{sign} \mathrm{d}f_a + \operatorname{sign} \mathrm{d}f_b = 0,$$

因此 (在所有这些弧上求和) $\deg(f; y) = 0$.

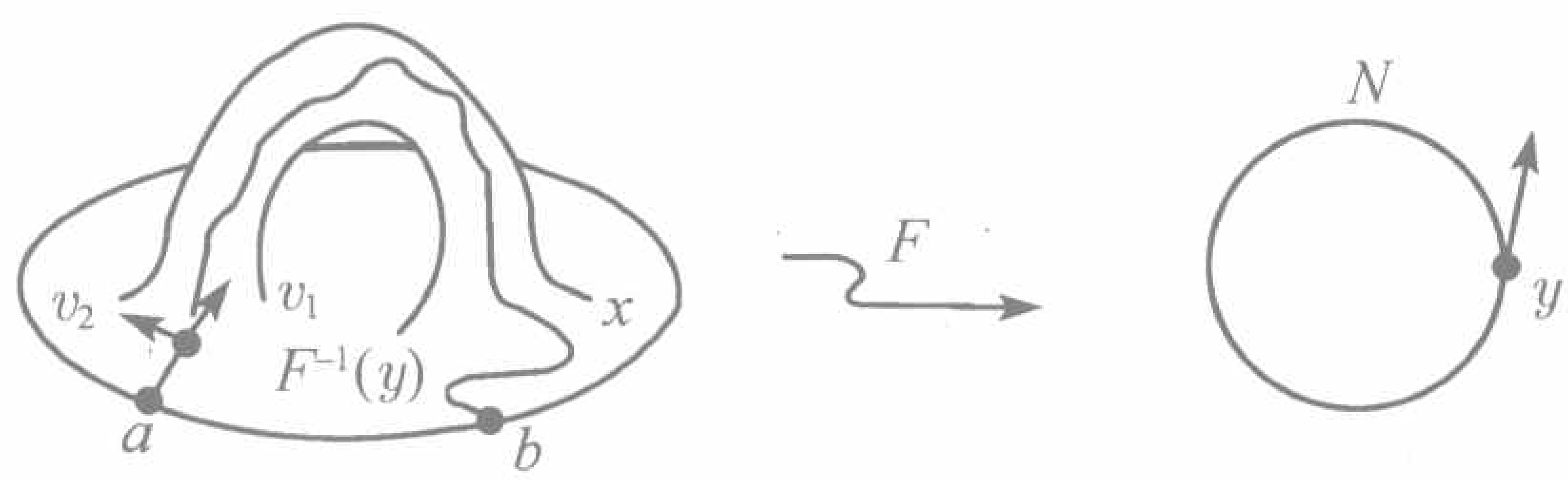


图 9 如何给 $F^{-1}(y)$ 定向

$X$ 的定向和 $N$ 的定向决定 $A$ 的一个定向如下: 给定 $x \in A$, 令 $(v_1, \cdots, v_{n+1})$ 为 $TX_x$ 的一个正的有向基, 其中 $v_1$ 与 $A$ 相切. 则 $v_1$ 决定的 $TA_x$ 的定向为所需要的定向当且仅当 $\mathrm{d}F_x$ 将 $(v_2, \cdots, v_{n+1})$ 变为 $TN_y$ 的正的有向基.

令 $v_1(x)$ 表示在点 $x$ 处与 $A$ 相切的正向单位向量. 显然 $v_1$ 是一个光滑函数, 并且 $v_1(x)$ 在一个边点 (设为 $b$) 上指向外, 而在另一个边点 $a$ 上指向内.

这立即推出

$$\operatorname{sign} \mathrm{d}f_a = -1, \quad \operatorname{sign} \mathrm{d}f_b = +1,$$

因此和为 0. 在所有这些弧 $A$ 上加起来, 便证明了 $\deg(f; y) = 0$.

更一般些, 设 $y_0$ 是 $f$ 的一个正则值但不是 $F$ 的正则值. 函数 $\deg(f; y)$ 在 $y_0$ 的某一邻域 $U$ 中为常数. 因此如在第 4 章中一样, 可

以在 $U$ 内取到一个 $F$ 的正则值 $y$. 从而

$$\deg(f; y_0) = \deg(f; y) = 0.$$

这就证明了引理 1.

现在考虑光滑映射 $f(x) = F(0, x)$ 与 $g(x) = F(1, x)$ 之间的光滑同伦 $F : [0, 1] \times M \to N$.

**引理 2**　对于任意共同的正则值 $y$, 度 $\deg(g; y)$ 等于 $\deg(f; y)$.

**证明**　流形 $[0, 1] \times M^n$ 能作为乘积给予定向, 并且它有由 $1 \times M^n$ (具有正确的定向) 与 $0 \times M^n$ (具有错误的定向) 组成的边界①. 于是在正则值 $y$ 处 $F \mid \partial([0, 1] \times M^n)$ 的度等于差

$$\deg(g; y) - \deg(f; y).$$

根据引理 1, 此差必定为零.

定理 A 与定理 B 的剩余部分的证明完全类似于第 4 章中的论证. 若 $y$ 与 $z$ 两者都是 $f : M \to N$ 的正则值, 选取一个将 $y$ 变为 $z$ 且同痕于恒同映射的微分同胚 $h : N \to N$. 则 $h$ 保持定向, 且经验证

$$\deg(f; y) = \deg(h \circ f; h(y)).$$

但 $f$ 同伦于 $h \circ f$, 因此根据引理 2,

$$\deg(h \circ f; z) = \deg(f; z),$$

从而 $\deg(f; y) = \deg(f; z)$. 定理证毕.

**例**　复函数 $z \to z^k$, $z \neq 0$, 将单位圆周映射到自身上, 其度为 $k$. (此处 $k$ 可为正数、负数或零). 退化映射

$$f : M \longrightarrow \text{常值} \in N$$

的度为零. 一个微分同胚 $f : M \to N$的度为 $+1$ 或 $-1$, 按 $f$ 保持或反转定向而定. 于是, 紧致的无边流形的反转定向的微分同胚不光滑同伦于恒同映射.

① “$1 \times M^n$ (具有正确的定向)” 意思是: 将 $M^n$ 的定向考虑作为 $1 \times M^n$ 的定向与由 $I \times M^n$ 决定的 $1 \times M^n$ 的定向相同; “$0 \times M^n$ (具有错误的定向)” 意思是: 将 $M^n$ 的定向考虑作为 $0 \times M^n$ 的定向与由 $I \times M^n$ 决定的 $0 \times M^n$ 的定向相反.
—— 译者注

反射 $r_i : S^n \to S^n$ 定义为

$$r_i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, -x_i, \cdots, x_{n+1}),$$

这是反转定向的微分同胚的例子. $S^n$ 的对径映射的度等于 $(-1)^{n+1}$, 只要注意这个映射为 $n+1$ 个反射的复合

$$-x = r_1 \circ r_2 \circ \cdots \circ r_{n+1}(x)$$

便可了解. 于是, 若 $n$ 是偶数, $S^n$ 的对径映射不光滑同伦于恒同映射, 这是一个用模 2 度不能判明的事实.

作为一个应用, 遵循 Brouwer, 我们证明: $S^n$ 容许有一个光滑的非零切向量场, 当且仅当 $n$ 是奇数. (参见图 10 和图 11.)

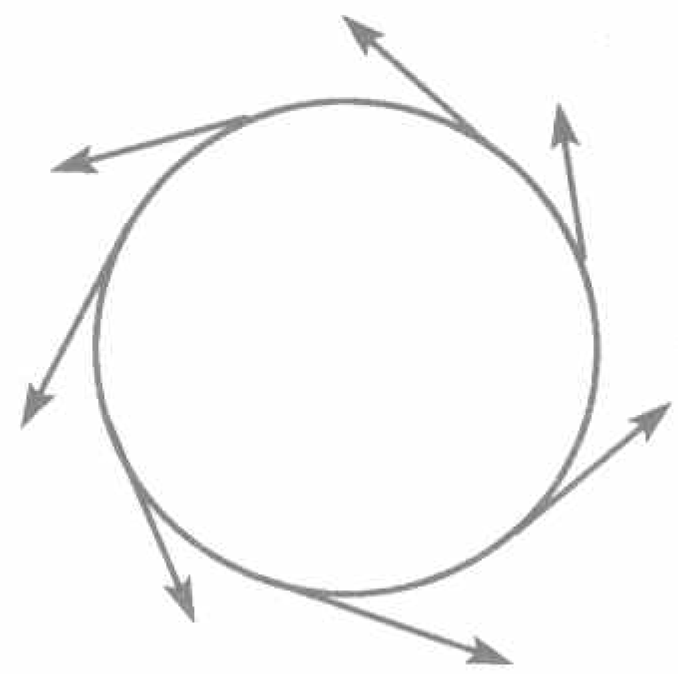

图 10 1 维球上的非零向量场

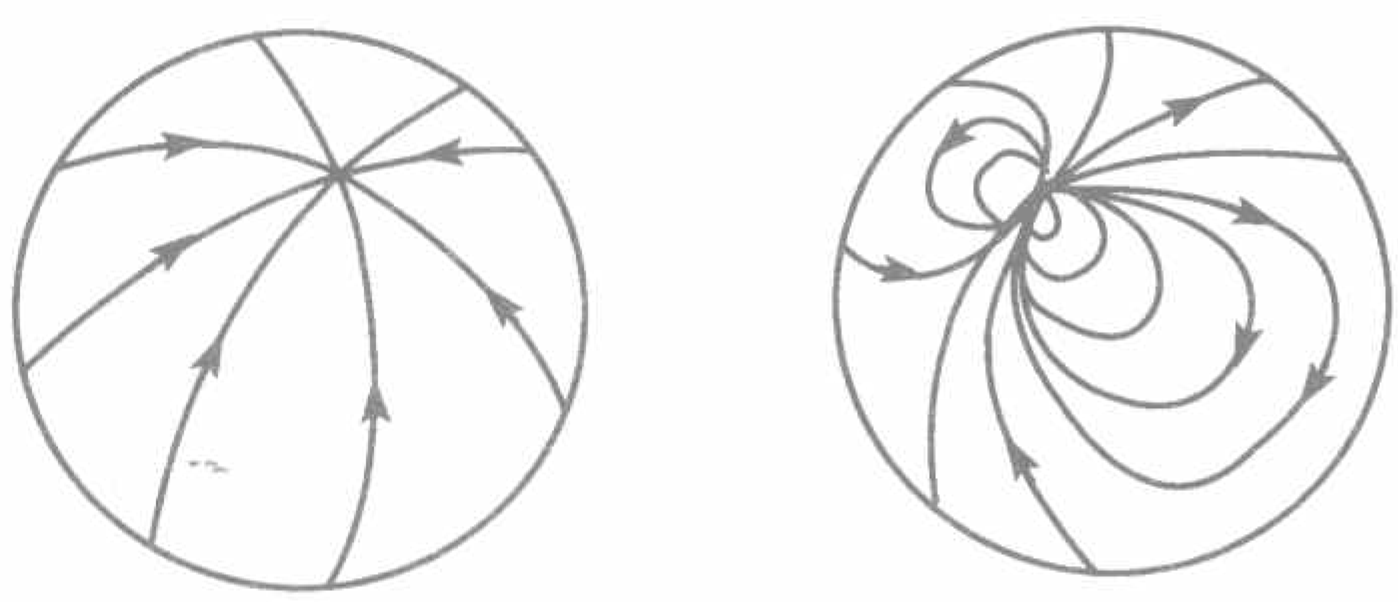

图 11 $n = 2$ 时的两个尝试

**定义** $M \subset \mathbf{R}^k$ 上的一个光滑的**切向量场** (tangent vector field) 乃是一个光滑映射 $v : M \to \mathbf{R}^k$, 满足条件: 对于每一个 $x \in M$, $v(x) \in TM_x$. 对于球面 $S^n \subset \mathbf{R}^{n+1}$ 的情形, 这明显地等价于条件:

$$\text{对于所有的 } x \in S^n,\ v(x) \cdot x = 0 \tag{1}$$

其中 "·" 表示欧氏内积.

若 $v(x)$ 对于所有 $x$ 都是非零的, 则也可假设:

$$\text{对于所有 } x \in S^n,\ v(x) \cdot v(x) = 1. \tag{2}$$

因为在任何情形下, $\overline{v}(x) = v(x)/\parallel v(x) \parallel$ 都将是满足这一条件的向量场. 于是可以认为 $v$ 是一个从 $S^n$ 到自身的光滑函数.

现在, 用公式 $F(x, \theta) = x\cos\theta + v(x)\sin\theta$ 定义一个光滑同伦

$$F : S^n \times [0, \pi] \longrightarrow S^n.$$

计算表明

$$F(x, \theta) \cdot F(x, \theta) = 1,$$

并且

$$F(x, 0) = x, \quad F(x, \pi) = -x.$$

于是 $S^n$ 的对径映射同伦于恒同映射. 当 $n$ 为偶数时, 我们已经知道这是不可能的.

另一方面, 若 $n = 2k - 1$, 解析表达式

$$v(x_1, \cdots, x_{2k}) = (x_2, -x_1, x_4, -x_3, \cdots, x_{2k}, -x_{2k-1})$$

在 $S^n$ 上确定一个非零切向量场. 证明完成.

这里顺便得到了: $n$ 为奇数时, $S^n$ 的对径映射同伦于恒同映射. Heinz Hopf 有一个著名定理: 从连通的 $n$ 维流形到 $n$ 维球面的两个映射是光滑同伦的, 当且仅当它们有相同的度. 在后面第 7 章中, 我们将证明一个蕴含着 Hopf 定理的更为一般的结果.

# 第 6 章　向量场与 Euler 数

进一步应用度的概念, 下面来研究在其他流形上的向量场.

首先考虑开集 $U \subset \mathbf{R}^m$ 和以点 $z \in U$ 为孤立零点的光滑向量场

$$v : U \longrightarrow \mathbf{R}^m.$$

函数

$$\overline{v} = v(x) / \| v(x) \|$$

将以 $z$ 为中心的小球面映射到单位球面中①. 这一映射的度称为 $v$ 在零点 $z$ 处的**指数**(index) $\iota$②.

图 12 说明了指数为 $-1, 0, 1, 2$ 的一些例子. (与 $v$ 有着密切联系的是这样一些“相切于” $v$ 的曲线, 它们可以通过解微分方程 $\mathrm{d}x_i/\mathrm{d}t =$

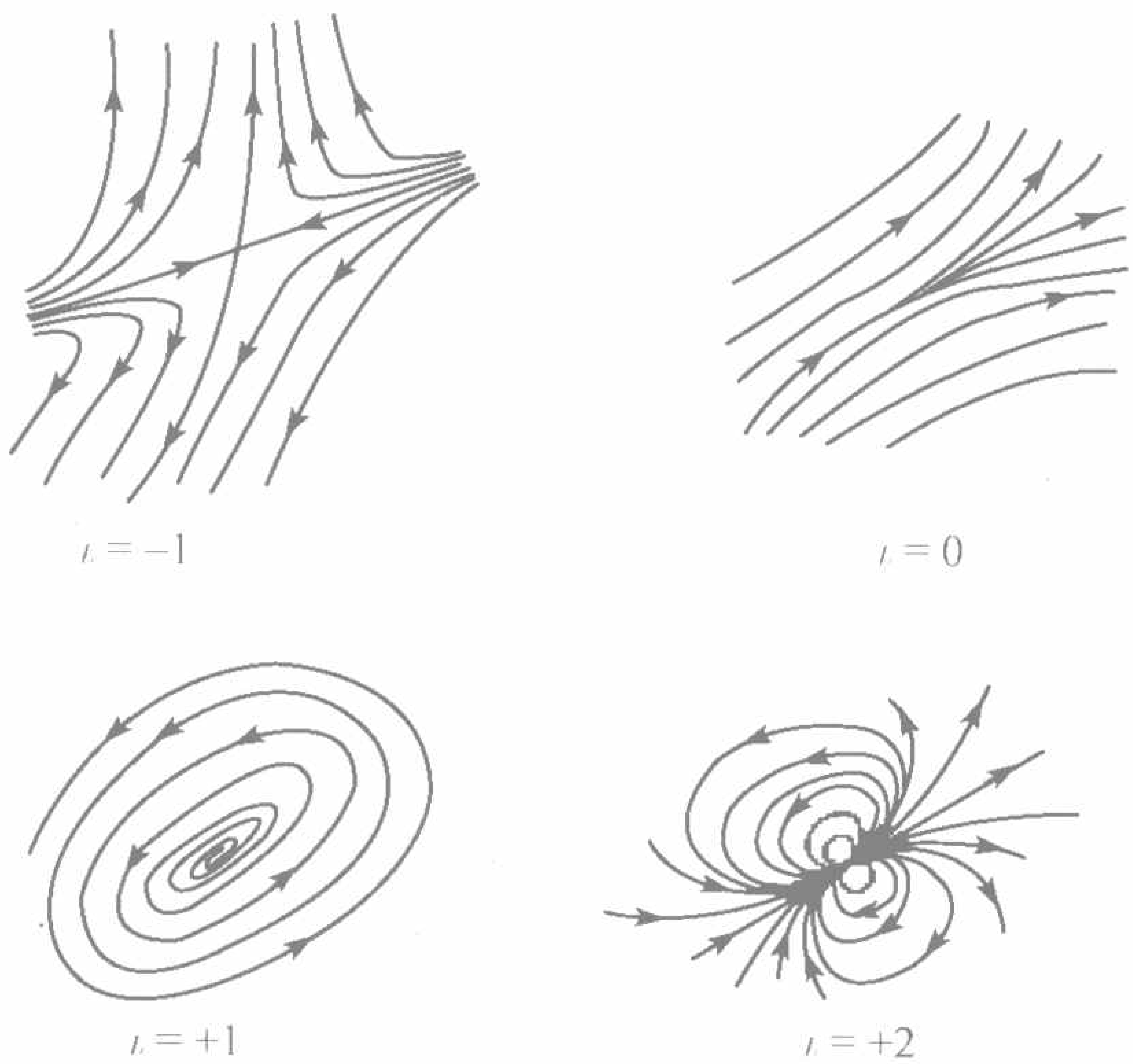


图 12　平面向量场的例子

① 每一个球面都作为相应的圆盘的边而予以定向.

② 在这一段指数的定义中所谈到的“小球”应具有如下的性质: 这小球连同它所包围的球体中无 $v$ 的其他零点, 并且应补充证明指数 $\iota$ 的定义与定义中涉及的小球的选择无关. —— 译者注

$v_i(x_1, \cdots, x_n)$ 而得到. 实际上画在图 12 中的便是这些曲线.)

具有任何一个指数的零点可按如下方式得到：在复数平面上，多项式 $z^k$ 确定一个光滑向量场，原点是它的指数为 $k$ 的零点；而函数 $\bar{z}^k$ 确定一个光滑向量场，原点是它的指数为 $-k$ 的零点.

我们必须证明指数的概念在 $U$ 的微分同胚下是不变的. 为了说清这层意思，考虑更为一般的情形，即映射 $f: M \to N$，且在 $M$ 和 $N$ 上各有一个向量场.

**定义**　如果对于每一个点 $x \in M$，$\mathrm{d}f_x$ 将 $v(x)$ 变为 $v'(f(x))$，则称流形 $M$ 上的向量场 $v$ 与流形 $N$ 上的向量场 $v'$ 在映射 $f$ 下是对应的.

若 $f$ 是一个微分同胚，则显然 $v'$ 被 $v$ 唯一地确定. 下面要用到记号

$$v' = \mathrm{d}f \circ v \circ f^{-1}.$$

**引理 1**　*设 $U$ 上的向量场 $v$ 在微分同胚 $f: U \to U'$ 下对应于 $U'$ 上的向量场*

$$v' = \mathrm{d}f \circ v \circ f^{-1}.$$

*则 $v$ 在孤立零点 $z$ 处的指数等于 $v'$ 在 $f(z)$ 处的指数.*

假定引理 1 成立，我们便能对任意流形 $M$ 上的向量场 $w$ 定义指数的概念如下：若 $g: U \to M$ 为 $M$ 中的孤立零点 $z$ 的某一邻域的参数化，则 $w$ 在点 $z$ 处的指数 $\iota$ 定义为等于 $U$ 上对应的向量场 $\mathrm{d}g^{-1} \circ w \circ g$ 在零点 $g^{-1}(z)$ 处的指数. 很明显，从引理 1 可知，$\iota$ 是完全确定的.

引理 1 的证明将建立在一个完全不同的结果的证明的基础上.

**引理 2**　*$\mathbf{R}^m$ 的任何一个保持定向的微分同胚 $f$ 光滑地同痕于恒同映射.*

(大不相同的是：对于许多 $m$ 的值，存在球 $S^m$ 的保持定向的微分同胚，这些微分同胚却不光滑地同痕于恒同映射. 见 [20, p.404].)

**证明**　可以假定 $f(0) = 0$. 因为在 0 点处的导射可定义为

$$\mathrm{d}f_0(x) = \lim_{t \to 0} f(tx)/t,$$

所以公式

$$F(x,t) = \begin{cases} f(tx)/t, & \text{当 } 0 < t \leqslant 1 \text{ 时;} \\ \mathrm{d}f_0(x), & \text{当 } t = 0 \text{ 时} \end{cases}$$

很自然地确定一个同痕

$$F : \mathbf{R}^m \times [0,1] \longrightarrow \mathbf{R}^m.$$

为证明 $F$ 即使当 $t \to 0$ 时也是光滑的, 将 $f$ 写成形式[①]

$$f(x) = x_1 g_1(x) + \cdots + x_m g_m(x),$$

其中 $g_1, \cdots, g_m$ 是适当的光滑函数, 并注意对于所有 $t$ 的值,

$$F(x,t) = x_1 g_1(tx) + \cdots + x_m g_m(tx).$$

于是 $f$ 同痕于线性映射 $\mathrm{d}f_0$, 而后者显然同痕于恒同映射. 这就证明了引理 2.

**引理 1 的证明**　可以假设 $z = f(z) = 0$ 以及 $U$ 为一个凸集. 若 $f$ 保持定向, 则与上面的过程一样, 构造一个单参数嵌入族

$$f_t : U \longrightarrow \mathbf{R}^m$$

使得满足 $f_0 =$ 恒同映射, $f_1 = f$, 并且对于所有的 $t$ 有 $f_t(0) = 0$. 令 $v_t$ 表示与 $U$ 上的向量场 $v$ 对应的 $f_t(U)$ 上的向量场 $\mathrm{d}f_t \circ v \circ f_t^{-1}$. 这些向量场都是完全确定的, 并且在一个以点 0 为中心的充分小的球上都是非零的. 因此 $v = v_0$ 在点 0 处的指数必定等于 $v' = v_1$ 在点 0 处的指数. 这就对于保持定向的微分同胚证明了引理 1.

为了考虑反转定向的微分同胚, 只要考虑反射 $\rho$ 这一特殊情形. 于是

$$v' = \rho \circ v \circ \rho^{-1},$$

故在 $\epsilon$ 球上的关联函数 $\overline{v}'(x) = v'(x)/\left\|v'(x)\right\|$ 满足条件

$$\overline{v}' = \rho \circ \overline{v} \circ \rho^{-1}.$$

显然 $\overline{v}'$ 的度等于 $\overline{v}$ 的度. 这便完成了引理 1 的证明.

现在研究下面这个经典结果: 令 $M$ 为一个紧致流形, $w$ 为 $M$ 上具有孤立零点的一个光滑向量场. 如果 $M$ 有边, 则要求 $w$ 在所有边点上都指向外.

① 例子可参见 [22, p.5].

**Poincaré-Hopf 定理** 在这样一个向量场上的所有零点处的指数和 $\sum \iota$ 等于 Euler 数[①]

$$\chi(M)=\sum_{i=0}^{m}(-1)^i[H_i(M)\text{ 的秩 }].$$

特别地, 这一指数和是 $M$ 的一个拓扑不变量: 它不依赖于向量场的特殊选取.

(这个定理的 2 维情形是由 Poincaré 在 1885 年证明的. 全部定理是 Hopf [14] 于 1926 年在 Brouwer 和 Hadamard 的较早的部分结果的基础上证明的.)

我们证明这个定理的一部分, 而约略勾划一下其余部分的证明. 首先考虑 $\mathbf{R}^m$ 中紧致区域这种特殊情形:

令 $X\subset\mathbf{R}^m$ 表示一个紧致的有边 $m$ 维流形. Gauss 映射 (Gauss mapping)

$$g:\partial X\longrightarrow S^{m-1}$$

对于每一个点 $x\in\partial X$ 指定了一个在点 $x$ 处的向外的单位法向量.

**引理 3 (Hopf)** 若 $v:X\to\mathbf{R}^m$ 为一个具有孤立零点的光滑向量场, 并且在边上 $v$ 指向 $X$ 的外面, 则指数和 $\sum\iota$ 等于从 $\partial X$ 到 $S^{m-1}$ 的 Gauss 映射的度. 特别地, $\sum\iota$ 不依赖于 $v$ 的选取.

例如, 若圆盘 $D^m$ 上的向量场在边上指向外面, 则 $\sum\iota=+1$. (参见图 13.)

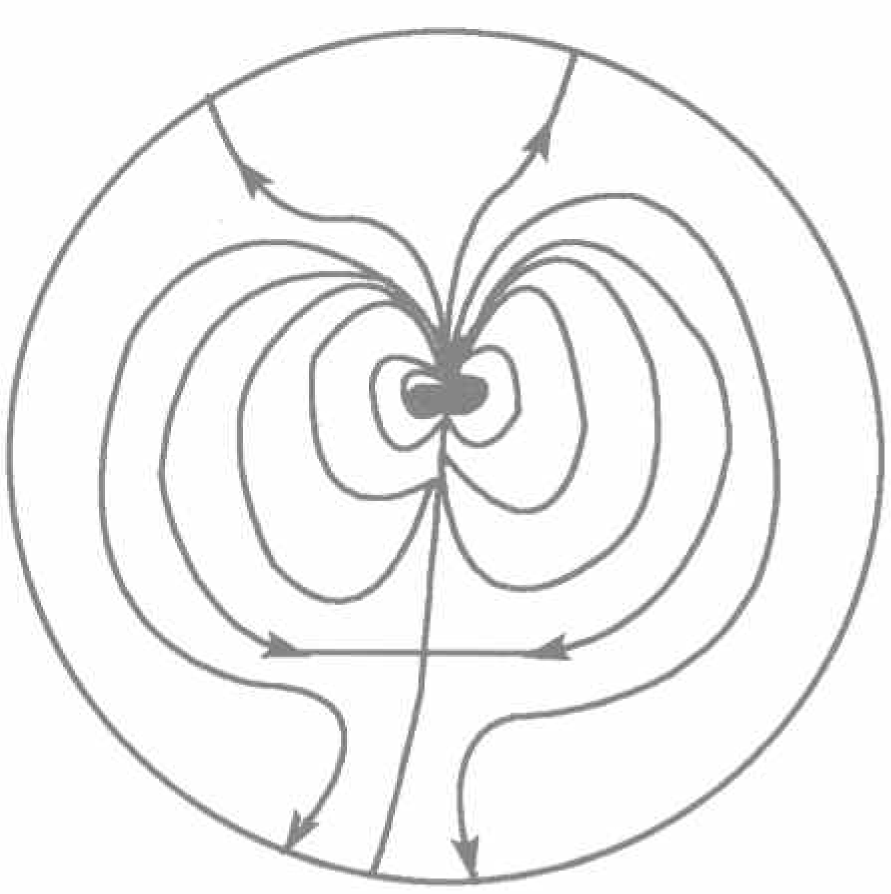

图 13 指数和为 +1 的例子

① 这里 $H_i(M)$ 表示 $M$ 的第 $i$ 个同调群. 这是我们第一次也是最后一次提到同调论.

**证明** 围绕着每一个零点挖去一个 $\epsilon$ 球体, 得到一个新的有边流形. 函数

$$\overline{v}(x) = v(x)/\,\|v(x)\|$$

将这一流形映射到 $S^{m-1}$ 中. 因此, $\overline{v}$ 限制于各个边的分支上的度的和为零. 但 $\overline{v} \mid \partial X$ 同伦于 $g$, 并且在其余的边的分支上的度加起来为 $-\sum \iota$ (负号出现是因为每一个小球都得到错误的定向①). 因此

$$\deg(g) - \sum \iota = 0.$$

正是我们需要证明的.

**注记** $g$ 的度也常作为 $\partial X$ 的"全曲率"而为人们所知晓, 这是因为它能被表为高斯曲率在 $\partial X$ 上的积分的常数倍. 这个整数无疑等于 $X$ 的 Euler 数. 当 $m$ 为奇数时, 它等于 $\partial X$ 的 Euler 数的一半.

在把这一结果推广到其他流形上以前, 我们需要某些进一步的准备知识.

自然希望能够借助于向量场 $v$ 在零点 $z$ 处的偏导数来计算 $v$ 在点 $z$ 处的指数. 首先考虑开集 $U \subset \mathbf{R}^m$ 上的向量场 $v$, 并且把 $v$ 看成是映射 $U \longrightarrow \mathbf{R}^m$, 所以 $\mathrm{d}v_z : \mathbf{R}^m \to \mathbf{R}^m$ 是有定义的.

**定义** 如果线性变换 $\mathrm{d}v_z$ 是非退化的, 则称向量场 $v$ 在点 $z$ 处是**非退化的** (nondegenerate).

因此, 非退化零点 $z$ 是孤立零点.

**引理 4** *$v$ 在非退化零点 $z$ 处的指数是 $+1$ 或是 $-1$, 按照 $\mathrm{d}v_z$ 的行列式是正的或是负的而定.*

**证明** 将 $v$ 看成一个从 $z$ 的某一凸邻域 $U_0$ 到 $\mathbf{R}^m$ 中的一个微分同胚. 可以假定 $z = 0$. 若 $v$ 保持定向, 我们已经知道了 $v \mid U_0$ 能够光滑地形变为恒同映射而不引入任何新的零点 (见引理 1 和引理 2). 因此, 指数肯定等于 $+1$.

若 $v$ 反转定向, 则类似地, $v$ 能形变为一个反射, 因此 $\iota = -1$.

更一般地, 考虑在流形 $M \subset \mathbf{R}^k$ 上的向量场 $w$ 的零点 $z$. 将 $w$ 看作一个从 $M$ 到 $\mathbf{R}^k$ 的映射, 所以导射 $\mathrm{d}w_z : TM_z \to \mathbf{R}^k$ 是有定义的.

① 在零点的指数的定义中, 每个小球的定向应作为圆盘 (即 $\epsilon$ 球体) 的边而得到. 而现在却作为除去了圆盘的其余部分的边而得到. 因此这两个定向恰好相反.

—— 译者注

**引理 5**　由于导射 $\mathrm{d}w_x$ 实际上将 $TM_z$ 变到子空间 $TM_z \subset \mathbf{R}^k$ 中, 因此可以把它看作是从 $TM_z$ 到自身的线性变换. 如果这个线性变换的行列式 $D \neq 0$, 则 $z$ 为 $w$ 的一个孤立零点, 它的指数为 $+1$ 或 $-1$, 按 $D$ 为正或为负而定.

**证明**　设 $h: U \to M$ 为 $z$ 的某一个邻域的参数化. 令 $e^i$ 表示 $\mathbf{R}^m$ 的第 $i$ 个基向量, 且令

$$t^i = \mathrm{d}h_u(e^i) = \partial h/\partial u_i,$$

从而向量 $t^1, \cdots, t^m$ 形成切空间 $TM_{h(u)}$ 的一个基. 现在需要计算 $t^i = t^i(u)$ 在线性变换 $\mathrm{d}w_{h(u)}$ 下的象. 首先注意

$$\mathrm{d}w_{h(u)}(t^i) = \mathrm{d}(w \circ h)_u(e^i) = \partial w(h(u))/\partial u_i. \tag{1}$$

设 $v = \sum v_j e^j$ 为 $U$ 上的、对应于 $M$ 上的向量场 $w$ 的那个向量场. 根据定义 $v = \mathrm{d}h^{-1} \circ w \circ h$, 于是

$$w(h(u)) = \mathrm{d}h_u(v) = \sum v_j t^j.$$

因此

$$\partial w(h(u))/\partial u_i = \sum_j (\partial v_j/\partial u_i) t^j + \sum_j v_j (\partial t^j/\partial u_i). \tag{2}$$

联合 (1) 和 (2), 并且在 $v$ 的零点 $h^{-1}(z)$ 处计算值, 则得到公式

$$\mathrm{d}w_z(t^i) = \sum_j (\partial v_j/\partial u_i) t^j. \tag{3}$$

于是 $\mathrm{d}w_z$ 映射 $TM_z$ 到自身中, 并且线性变换 $TM_z \longrightarrow TM_z$ 的行列式 $D$ 等于矩阵 $(\partial v_j/\partial''_i)$ 的行列式. 结合引理 4 即得本引理的证明.

现在考虑紧致的无边流形 $M \subset \mathbf{R}^k$. 令 $N_\epsilon$ 表示 $M$ 的闭的 $\epsilon$ 邻域 (即满足条件: 对于某一 $y \in M$ 使 $\|x - y\| \leqslant \epsilon$ 的所有 $x \in \mathbf{R}^k$ 的集合). 当 $\epsilon$ 充分小时, 能够证明 $N_\epsilon$ 是光滑的有边流形. (见第 8 章, 问题 11.)

**定理**　对于 $M$ 上只有非退化零点的任意向量场 $v$, 指数和 $\sum \iota$ 等于 Gauss 映射

$$g: \partial N_\epsilon \longrightarrow S^{k-1}$$

的度[①]. 特别地, 这一指数和是不依赖于向量场的选取的.

**证明**　对于 $x \in N_\epsilon$, 设 $r(x) \in M$ 表示 $M$ 中最接近于 $x$ 的点 (参见第 8 章, 问题 12). 注意这时向量 $x - r(x)$ 是正交于 $M$ 在点 $r(x)$ 处的切空间的, 否则, $r(x)$ 将不是 $M$ 中最接近于 $x$ 的点. 若 $\epsilon$ 充分小, 则映射 $r(x)$ 必是光滑的, 并且是完全确定的.

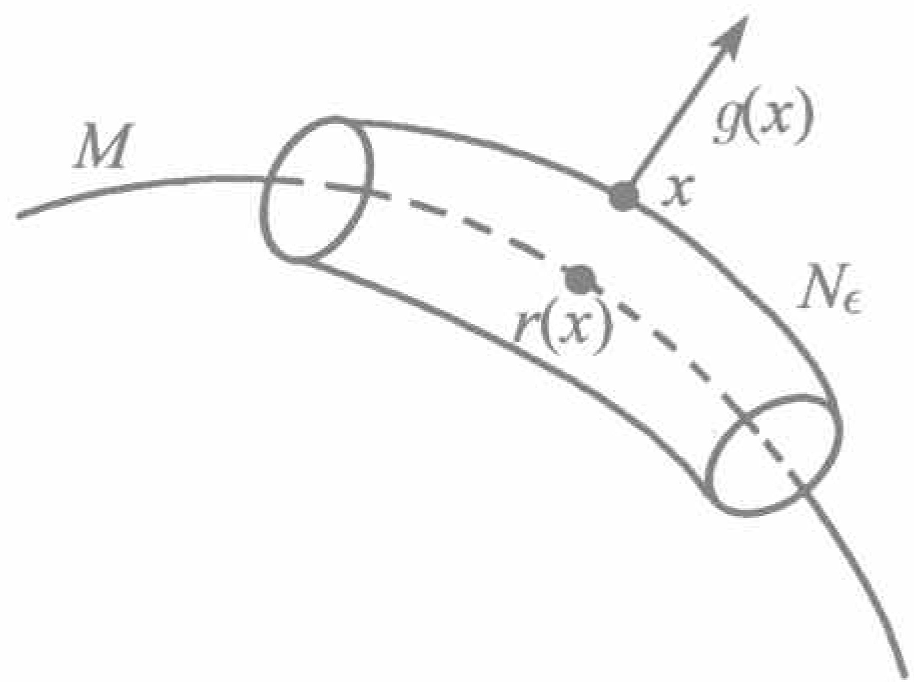


图 14　$M$ 的 $\epsilon$ 邻域

现在还要考虑距离平方函数

$$\varphi(x) = \| x - r(x) \|^2 .$$

简单的计算表明, $\varphi$ 的梯度由

$$\operatorname{grad} \varphi = 2(x - r(x))$$

给出. 因此, 对于等高曲面 $\partial N_\epsilon = \varphi^{-1}(\epsilon^2)$ 的每一个点 $x$, 向外的单位法向量由

$$g(x) = \operatorname{grad} \varphi / \| \operatorname{grad} \varphi \| = (x - r(x))/\epsilon$$

给出. 将 $v$ 扩充为邻域 $N_\epsilon$ 上的向量场 $w$, $w$ 的定义是

$$w(x) = (x - r(x)) + v(r(x)).$$

则 $w$ 在边上指向外, 因为内积 $w(x) \cdot g(x)$ 等于 $\epsilon > 0$. 注意仅在 $M$ 中 $v$ 的零点处 $w$ 才能为零, 这是显然的, 因为两个加项 $(x - r(x))$ 与 $v(r(x))$ 是互相正交的. 在零点 $z \in M$ 处计算 $w$ 的偏导数, 可以看到

$$dw_z(h) = \begin{cases} dv_z(h), & \text{对于所有 } h \in TM_z; \\ h, & \text{对于 } h \in TM_z^{\perp}. \end{cases}$$

① Allendoerfer 与 Fenchel 给了此度一个不同的解释: $g$ 的度能够表示为某一适当的曲率量在 $M$ 上的积分, 于是便产生了经典的 Gauss-Bonnet 定理的 $m$ 维推广. (参考 [1]、[9]. 也可参见 Chern [6].)

于是 $dw_z$ 的行列式等于 $dv_z$ 的行列式. 因此 $w$ 在零点 $z$ 处的指数等于 $v$ 在 $z$ 处的指数 $\iota$.

现在根据引理 3 可见, 指数和 $\sum \iota$ 等于 $g$ 的度. 这就证明了定理 1.

**例** 在球 $S^m$ 上存在一个向量场 $v$, 它在每一个点处都指向"北"[1]. 在南极处各向量向外放射, 因此指数是 $+1$; 在北极处各向量向内汇集, 因此指数是 $(-1)^m$. 于是不变量 $\sum \iota$ 等于 0 或 2 按 $m$ 是奇数或偶数而定. 这就重新证明了在偶维球上的每一个向量场都有零点.

对于任意奇维的无边流形, 不变量 $\sum \iota$ 为零. 因为如果把向量场 $v$ 换成 $-v$, 则每一个指数都是乘了 $(-1)^m$. 从而等式

$$\sum \iota = (-1)^m \sum \iota.$$

当 $m$ 为奇数时, 这蕴含 $\sum \iota = 0$.

**注记** 若在一个连通流形 $M$ 上 $\sum \iota = 0$, 则 Hopf 的定理断定 $M$ 上存在一个根本没有零点的向量场.

为了完成 Poincaré-Hopf 定理的证明, 以下 3 个步骤是必要的.

**第 1 步** 不变量 $\sum \iota$ 与 Euler 数 $\chi(M)$ 恒等. 这只需适当地构造一个在 $M$ 上的, 使得 $\sum \iota$ 与 $\chi(M)$ 相等的非退化向量场的例子即可. 作好这件事的一种很顺利的方法如下: 根据 M. Morse 的理论, 总可以找到 $M$ 上的一个实值函数, 它的梯度是一个非退化的向量场. 同时 Morse 证明了与这个梯度场相关的指数和等于 $M$ 的 Euler 数. 关于这一论证的详情, 请读者参考 Milnor[22, pp.29,36].

**第 2 步** 对于有退化零点的向量场, 证明这个定理. 首先考虑开集 $U$ 上的有一个孤立零点 $z$ 的向量场 $v$. 若

$$\lambda : U \longrightarrow [0, 1]$$

在 $z$ 的一个小邻域 $N_1$ 取值为 1, 而在一个稍大一点的邻域 $N$ 外取值为 0, 同时若 $y$ 是 $v$ 的一个足够小的正则值, 则向量场

$$v'(x) = v(x) - \lambda(x)y$$

---

[1] 例如: $v$ 可用公式 $v(x) = p - (p \cdot x)x$ 来定义, 其中 $p$ 是北极. (见图 11.)

在 $N$ 中是非退化的[①]. 在 $N$ 中的零点的指数和能够用映射

$$\overline{v}:\partial N \longrightarrow S^{m-1}$$

的度来计算, 因此, 在这种替换之下是不变的.

更一般地, 考虑紧致流形 $M$ 上的向量场. 局部地应用这一断言, 可以看到任意有孤立零点的向量场可被替换成一个非退化的向量场而不改变其指数和 $\sum\iota$.

**第 3 步**　有边流形. 若 $M \subset \mathbf{R}^k$ 有边, 则在边 $\partial M$ 上指向外面的任何一个向量场 $v$ 也能扩充到邻域 $N\epsilon$ 上, 并使其在 $\partial N_\epsilon$ 上指向外面. 不过在 $M$ 的边的周围的光滑性问题上存在着某些困难, 因为 $N\epsilon$ 不是光滑的 (即 $C^\infty$ 类可微的) 流形, 而仅仅是 $C^1$ 流形. 如果像前面一样, 用 $w(x) = v(r(x)) + x - r(x)$ 来定义扩展 $w$, 它在 $\partial M$ 附近将仅仅是连续向量场. 不过, 这一论断是能够证明的, 而不必给出更强的可微性假定. 此外还有其他的证明方法.

① 显然, 在 $N_1$ 中 $v'$ 是非退化的. 若 $y$ 充分小, 则 $v'$ 在 $N - N_1$ 中根本没有零点.

# 第 7 章　标架式协边和 Pontryagin 构造

映射 $M \longrightarrow M'$ 的度仅当流形 $M$ 和 $M'$ 是有定向的并且具有同样的维数时才有定义. 现在研究由 Pontryagin 给出的一种推广, 它对于任意从紧致无边流形到球的光滑映射

$$f : M \longrightarrow S^p$$

都有定义. 下面首先叙述某些定义.

令 $N$ 与 $N'$ 为 $M$ 的紧致的 $n$ 维子流形, 并且 $\partial N = \partial N' = \partial M = \varnothing$. 维数差 $m-n$ 称为这些子流形的 **余维数** (codimension).

**定义**　如果 $M \times [0,1]$ 的子集

$$N \times [0, \epsilon) \cup N' \times (1-\epsilon, 1]$$

能够扩张为一个紧致流形

$$X \subset M \times [0,1],$$

使得

$$\partial X = N \times 0 \cup N' \times 1,$$

并且使得 $X$ 与 $M \times 0 \cup M \times 1$ 的交点都是 $\partial X$ 的点, 则称 $N$ **在** $M$ **中协边于** (cobordant to) $N'$.

显然, 协边是一个等价关系. (见图 15.)

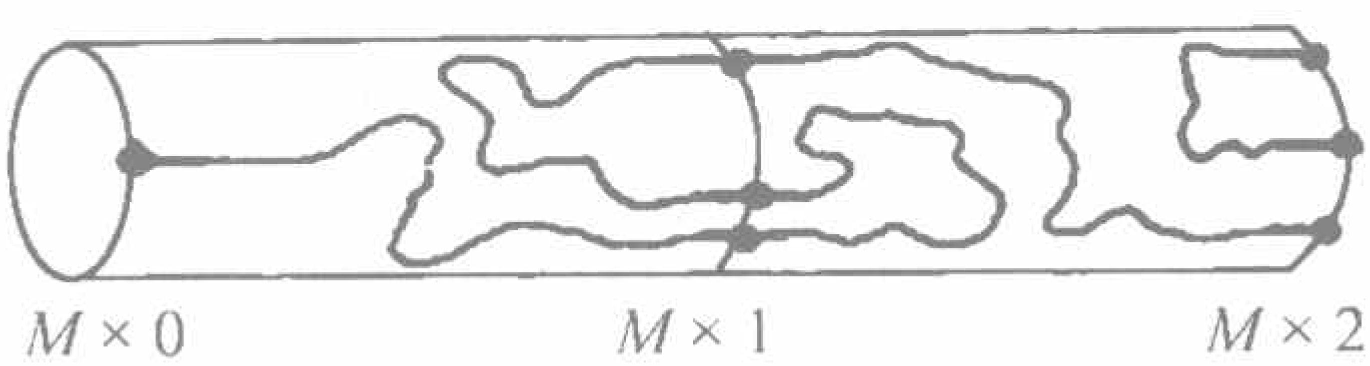


图 15　$M$ 中两个协边的粘接

**定义**　子流形 $N \subset M$ 的一个 **标架** (framing) 乃是一个光滑函数 $\mathfrak{v}$, 它对于每一个点 $x \in N$ 指定了 $M$ 中 $N$ 的法向量空间 $TN_x^\perp \subset TM_x$ 的一个基

$$\mathfrak{b}(x) = (v^1(x), \cdots, v^{m-n}(x)).$$

(见图 16.) 偶对 $(N, \mathfrak{b})$ 称为 $M$ 的**标架式子流形** (framed submanifold). 两个标架式子流形 $(N, \mathfrak{b})$ 与 $(N', \mathfrak{w})$ 是**标架式协边的** (framed cobordant), 如果存在 $N$ 与 $N'$ 间的一个协边 $X \subset M \times [0,1]$ 以及 $X$ 的一个标架 $\mathfrak{u}$, 使得

$$u^i(x,t) = \begin{cases} (v^i(x), 0), & \text{当 } (x,t) \in N \times [0,\epsilon) \text{ 时;} \\ (w^i(x), 0), & \text{当 } (x,t) \in N' \times (1-\epsilon, 1] \text{ 时.} \end{cases}$$

这也是一个等价关系.

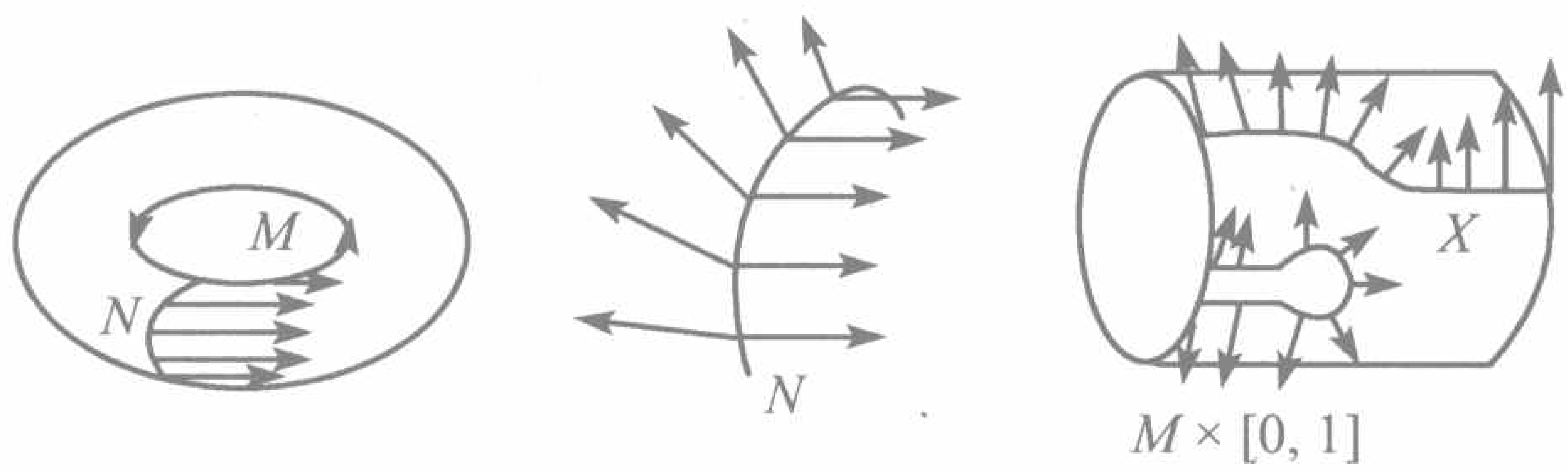


图 16　标架式子流形与标架式协边

现在考虑光滑映射 $f : M \to S^p$ 以及其正则值 $y \in S^p$. 映射 $f$ 诱导出流形 $f^{-1}(g)$ 的一个标架如下: 对于切空间 $T(S^p)_y$, 选取一个正的有向基 $\mathfrak{b} = (v^1, \cdots, v^p)$. 对于每一个 $x \in f^{-1}(y)$, 回顾第 2 章引理 2,

$$\mathrm{d}f_x : TM_x \longrightarrow T(S^p)_y$$

将子空间 $Tf^{-1}(y)_x$ 映射成零, 且将它的正交补空间 $Tf^{-1}(y)_x^{\perp}$ 同构地映射到 $T(S^p)_y$ 上. 因此存在唯一的一个向量

$$w^i(x) \in Tf^{-1}(y)_x^{\perp} \subset TM_x$$

在 $\mathrm{d}f_x$ 下映为 $v^i$. 为了方便, 将得到的 $f^{-1}(y)$ 的标架 $w^1(x), \cdots, w^p(x)$, 记作 $\mathfrak{w} = f^*\mathfrak{b}$.

**定义**　标架式流形 $(f^{-1}(y), f^*\mathfrak{b})$ 称为与 $f$ 相联系的**Pontryagin 流形** (Pontryagin manifold).

自然, 对于 $y$ 和 $\mathfrak{b}$ 的不同选择, $f$ 有许多 Pontryagin 流形, 但它们都属于同一个标架式协边类.

**定理 A**　*若 $y'$ 是 $f$ 的另一个正则值, 且 $\mathfrak{b}'$ 是 $T(S^p)_{y'}$ 的一个正的定向基, 则标架式流形 $(f^{-1}(y'), f^*\mathfrak{b}')$ 标架式协边于 $(f^{-1}(y), f^*\mathfrak{b})$.*

**定理 B** 从 $M$ 到 $S^p$ 的两个映射是光滑同伦的, 当且仅当与它们相联系的 Pontryagin 流形是标架式协边的.

**定理 C** $M$ 中任何一个余维数为 $p$ 的紧致的标架式子流形 $(N, \mathfrak{n})$ 均可认作某一光滑映射 $f: M \to S^p$ 的 Pontryagin 流形.

于是映射的同伦类便与子流形的标架式协边类一一对应.

定理 A 的证明非常类似于第 4 章与第 5 章中诸论断的证明, 它基于下列 3 个引理:

**引理 1** 若 $\mathfrak{v}$ 与 $\mathfrak{v}'$ 为在 $y$ 处的两个不同的正定向基, 则 Pontryagin 流形 $(f^{-1}(y), f^*\mathfrak{v})$ 标架式协边于 $(f^{-1}(y), f^*\mathfrak{v}')$.

**证明** 在 $T(S^p)_y$ 的所有正定向基的空间中选取一条从 $\mathfrak{v}$ 到 $\mathfrak{v}'$ 的光滑道路. 这是可能的, 因为这一基的空间能够等同于具有正行列式的矩阵的空间 $GL^+(p, R)$, 因此是连通的. 由这一道路引出所求的协边 $f^{-1} \times [0,1]$ 的标架.

粗略一点, 我们可常常在 "标架式流形 $(f^{-1}(y), f^*\mathfrak{v})$" 中省略 $f^*\mathfrak{v}$ 而简称 "标架式流形 $f^{-1}(y)$".

**引理 2** 若 $y$ 为 $f$ 的一个正则值, 且 $z$ 充分接近于 $y$, 则 $f^{-1}(z)$ 标架式协边于 $f^{-1}(y)$.

**证明** 因为临界值的集合 $f(C)$ 是紧致的, 故可以选到 $\epsilon > 0$ 使得 $y$ 的 $\epsilon$ 邻域只包含正则值. 给定 $z$ 满足 $\|z - y\| < \epsilon$, 选取一个光滑的单参数旋转族 (如, 同痕) $r_i: S^p \to S^p$, 使得 $r_1(y) = z$, 并且

(1) 当 $0 \leqslant t < \epsilon'$ 时, $r_t$ 为恒同映射;

(2) 当 $1 - \epsilon' < t \leqslant 1$ 时, $r_t$ 等于 $r_1$;

(3) 每一个 $r_t^{-1}(z)$ 都位于从 $y$ 到 $z$ 的大圆上, 因此为 $f$ 的正则值.

定义同伦
$$F: M \times [0,1] \longrightarrow S^p$$
为 $F(x,t) = r_t f(x)$. 对于每一个 $t$, 注意 $z$ 为复合映射
$$r_t \circ f: M \longrightarrow S^p$$
的正则值. 这就可推知 $z$ 为映射 $F$ 的正则值了. 于是
$$F^{-1}(z) \subset M \times [0,1]$$
为一个标架式流形, 并且它给出了标架式流形 $f^{-1}(z)$ 与 $(r_1 \circ f)^{-1}(z) = f^{-1} r_1^{-1}(z) = f^{-1}(y)$ 之间的一个标架式协边. 这就证明了引理 2.

**引理 3**　若 $f$ 与 $g$ 是光滑同伦的, 且 $y$ 是两者的正则值, 则 $f^{-1}(y)$ 标架式协边于 $g^{-1}(y)$.

**证明**　选取一个同伦 $F$, 满足

$$F(x,t)=\begin{cases} f(x), & \text{当 } 0\leqslant t<\epsilon \text{ 时;}\\ g(x), & \text{当 } 1-\epsilon<t\leqslant 1 \text{ 时.}\end{cases}$$

选取 $F$ 的一个足够靠近 $y$ 的正则值 $z$, 因而 $f^{-1}(z)$ 标架式协边于 $f^{-1}(y)$, 并且 $g^{-1}(z)$ 标架式协边于 $g^{-1}(y)$. 从而 $F^{-1}(z)$ 为一个标架式流形, 并且它给出 $f^{-1}(z)$ 和 $g^{-1}(z)$ 之间的一个标架式协边. 这就证明了引理 3.

**定理 A 的证明**　给定 $f$ 的任意两个正则值 $y$ 与 $z$, 我们能够选取一个光滑的单参数旋转族

$$r_t : S^p \longrightarrow S^p$$

使得 $r_0$ 为恒同, 并且 $r_1(y)=z$. 于是 $f$ 同伦于 $r_1\circ f$, 因此 $f^{-1}(z)$ 标架式协边于

$$(r_1\circ f)^{-1}(z)=f^{-1}r_1^{-1}(z)=f^{-1}(y).$$

这就完成了定理 A 的证明.

定理 C 的证明基于下列事实: 令 $N\subset M$ 为带着标架 $\mathfrak{v}$ 的, 余维数为 $p$ 的标架式子流形. 设 $N$ 是紧致的并且 $\partial N=\partial M=\varnothing$.

**乘积邻域定理**　$N$ 在 $M$ 中的某一邻域微分同胚于乘积 $N\times\mathbf{R}^p$. 此外, 微分同胚能够取得使每一个 $x\in N$ 对应着 $(x,0)\in N\times\mathbf{R}^p$, 并且使每一个法标架 $\mathfrak{v}(x)$ 对应于 $\mathbf{R}^p$ 的标准基.

**注记**　对于任意子流形, 乘积邻域不存在. (参见图 17.)

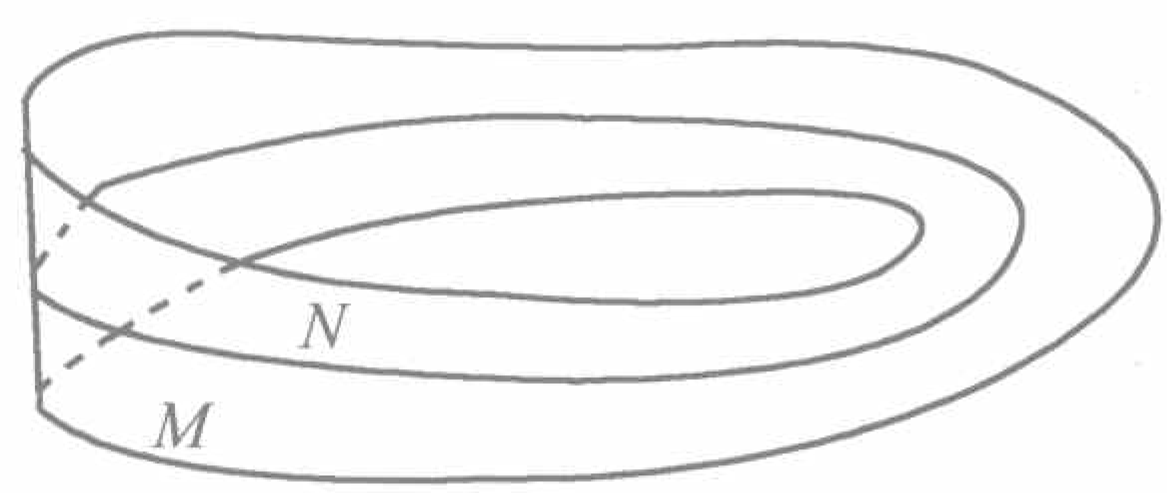


图 17　不可标架的子流形

**证明**　先设 $M$ 为欧氏空间 $\mathbf{R}^{n+p}$. 考虑映射 $g: N\times\mathbf{R}^p\longrightarrow M$ 其定义为

$$g(x;t_1,\cdots,t_p)=x+t_1v^1(x)+\cdots+t_pv^p(x).$$

显然 $\mathrm{d}g(x;0,\cdots,0)$ 是非退化的, 因此 $g$ 将 $(x,0)\in N\times\mathbf{R}^p$ 的某邻域微分同胚地映射到一个开集上.

如果 $\epsilon>0$ 充分小, 我们将证明 $g$ 在 $N\times 0$ 的整个邻域 $N\times U_\epsilon$ 上是一一的, 其中 $U_\epsilon$ 表示 0 在 $\mathbf{R}^p$ 中的 $\epsilon$ 邻域. 因为否则, 在 $N\times\mathbf{R}^p$ 中将有点偶 $(x,u)\neq(x',u')$, 其中 $\|u\|$ 与 $\|u'\|$ 任意小, 使得

$$g(x,u)=g(x',u').$$

由于 $N$ 是紧致的, 故可选到一个这种点偶的序列, 其中 $x$ 收敛于 $x_0$, $x'$ 收敛于 $x'_0$, 且 $u\to 0$, $u'\to 0$. 显然 $x_0=x'_0$, 而这与 $g$ 在 $(x_0,0)$ 的一个邻域中是一一的相矛盾.

于是 $g$ 将 $N\times U_\epsilon$ 微分同胚地映射到一个开集上, 但 $U_\epsilon$ 在对应

$$u\longrightarrow u/(1-\|u\|^2/\epsilon^2)$$

之下微分同胚于整个欧氏空间 $\mathbf{R}^p$. 因为 $g(x,0)=x$, 且因为 $\mathrm{d}g(x,0)$ 满足定理的要求, 这就对于 $M=\mathbf{R}^{n+p}$ 的特殊情形证明了乘积邻域定理.

对于一般情形, 必需将 $\mathbf{R}^{n+p}$ 中的直线换成 $M$ 中的测地线. 更精确地说, 令 $g(x;t_1,\cdots,t_p)$ 为 $M$ 中长度为 $\|t_1v^1(x)+\cdots+t_pv^p(x)\|$ 的测地线段的端点, 这个测地线段从 $x$ 开始, 且初速度向量为

$$t_1v^1(x)+\cdots+t_pv^p(x)/\|t_1v^1(x)+\cdots+t_pv^p(x)\|$$

熟悉测地线的读者, 不难验证

$$g:N\times U_\epsilon\longrightarrow M$$

当 $\epsilon$ 充分小时完全确定, 并且是光滑的. 余下的证明与前面相同.

**定理 C 的证明**　设 $N\subset M$ 为一个紧致的、无边的标架式子流形. 对于 $N$ 的邻域 $V$ 用上面的方法选取一个乘积表示

$$g:N\times\mathbf{R}^p\longrightarrow V\subset M,$$

并且定义投射

$$\pi:V\longrightarrow\mathbf{R}^p$$

为 $\pi(g(x,y))=y$. (见图 18.) 显然 0 为一个正则值, 且 $\pi^{-1}(0)$ 恰好是附有给定标架的 $N$.

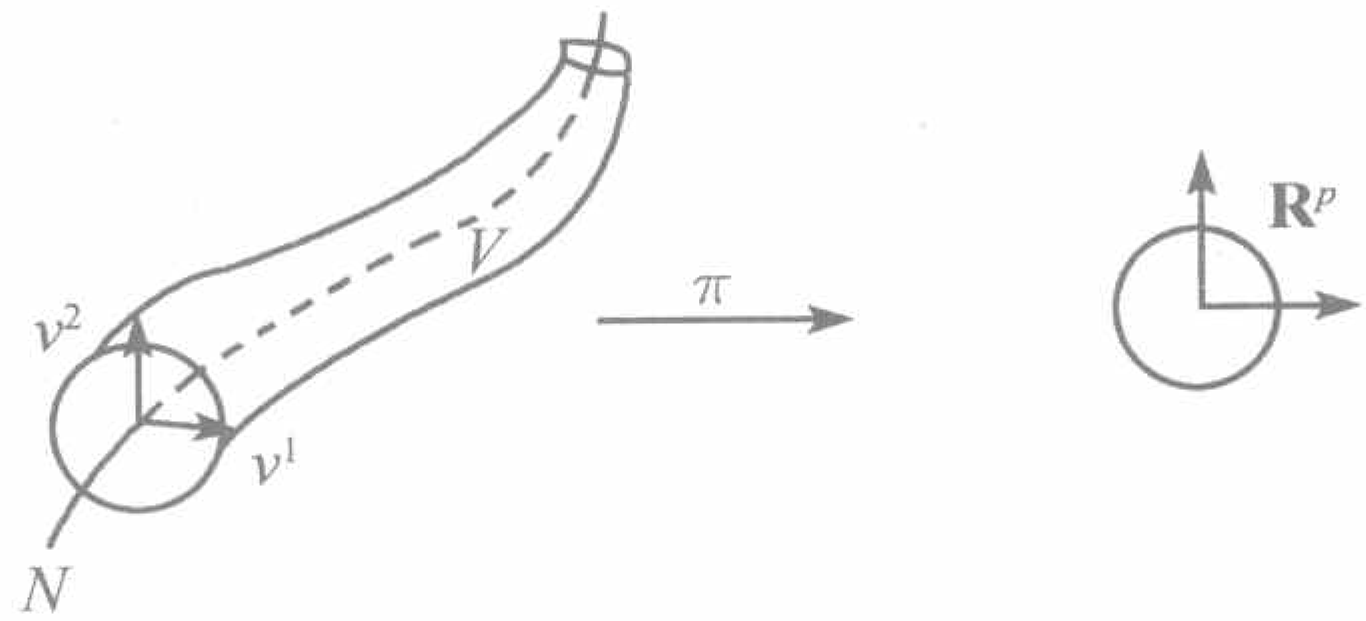


图 18　构造一个具有给定 Pontryagin 流形的映射

现在选取一个光滑映射 $\varphi:\mathbf{R}^p\to S^p$, 它把每一个满足 $\|x\|\geqslant 1$ 的点 $x$ 映射为一个基本点 $s_0$, 并且将 $\mathbf{R}^p$ 中的开单位球体微分同胚地[①]映射到 $S^p-s_0$. 定义

$$f:M\to S^p$$

为

$$f(x)=\begin{cases}\varphi(\pi(x)), & \text{当 } x\in V \text{ 时};\\ s_0, & \text{当 } x\notin V \text{ 时}.\end{cases}$$

显然 $f$ 是光滑的, 且点 $\varphi(0)$ 是 $f$ 的正则值. 由于相应的 Pontryagin 流形

$$f^{-1}(\varphi(0))=\pi^{-1}(0)$$

正好等于标架式流形 $N$, 这就完成了引理 C 的证明.

为了证明定理 B, 必需首先证明映射的 Pontryagin 流形决定它的同伦类. 令 $f,g:M\to S^p$ 表示具有共同的正则值 $y$ 的一个光滑映射.

**引理 4**　若标架式流形 $(f^{-1}(y),f^*\mathfrak{b})$ 等于标架式流形 $(g^{-1}(y),g^*\mathfrak{b})$, 则 $f$ 光滑同伦于 $g$.

**证明**　为方便计, 令 $N=f^{-1}(y)$. 假设 $f^*\mathfrak{b}=g^*\mathfrak{b}$ 意味着对于所有的 $x\in N$, $\mathrm{d}f_x=\mathrm{d}g_x$.

先设 $f$ 与 $g$ 在 $N$ 的某一邻域 $V$ 上完全重合. 令 $h:S^p-y\to\mathbf{R}^p$ 为球极平面投影. 那么, 由同伦

① 例如, $\varphi(x)=h^{-1}(x/\lambda(\|x\|^2))$, 其中 $h$ 为从 $s_0$ 出发的球极平面投影, $\lambda$ 为单调递减的光滑函数, 且满足条件: 当 $t<1$ 时, $\lambda(t)>0$; 当 $t\geqslant 1$ 时, $\lambda(t)=0$.

$$H(x,t)=\begin{cases} f(x), & \text{当 } x\in V \text{ 时;}\\ h^{-1}[t\cdot h(f(x))+(1-t)\cdot h(g(x))], & \text{当 } x\in M-N \text{ 时}. \end{cases}$$

可以知道: $f$ 光滑同伦于 $g$.

于是只要将 $f$ 变形, 使得在 $N$ 的某一小邻域上与 $g$ 重合, 但在变形的过程中必须注意, 不要把任何新的点都映射为 $y$. 对于 $N$ 的邻域 $V$ 选取一个乘积表示

$$N\times\mathbf{R}^p\longrightarrow V\subset M,$$

其中 $V$ 足够小, 使得 $f(V)$ 与 $g(V)$ 不包含 $y$ 的对径点 $\bar{y}$. 将 $V$ 与 $N\times\mathbf{R}^p$ 等同起来, 并且将 $S^p-\bar{y}$ 与 $\mathbf{R}^p$ 等同起来, 得到相应的映射

$$F,G:N\times\mathbf{R}^p\longrightarrow\mathbf{R}^p,$$

其中

$$F^{-1}(0)=G^{-1}(0)=N\times 0,$$

并且对于所有的 $x\in N$,

$$\mathrm{d}F_{(x,0)}=\mathrm{d}G_{(x,0)}=(\text{到 }\mathbf{R}^p\text{ 的投射}).$$

首先找一个常数 $c$, 使当 $x\in N$ 以及 $0<\|u\|<c$ 时,

$$F(x,u)\cdot u>0,\quad G(x,u)\cdot u>0.$$

这就是说, 点 $F(x,u)$ 与点 $G(x,u)$ 属于 $\mathbf{R}^p$ 中同一个开的半空间. 所以 $F$ 与 $G$ 之间的同伦

$$(1-t)F(x,u)+tG(x,u)$$

至少当 $\|u\|<c$ 时不会把任何新的点映射为 0.

根据 Taylor 定理, 当 $\|u\|\leqslant 1$ 时,

$$\|F(x,u)-u\|\leqslant c_1\|u\|^2.$$

因此

$$|(F(x,u)-u)\cdot u|\leqslant c_1\|u\|^3,$$

并且当 $0<\|u\|<c=\mathrm{Min}(c_1^{-1},1)$ 时,

$$F(x,u)\cdot u\geqslant\|u\|^2-c_1\|u\|^3>0.$$

对于 $G$ 有类似的不等式.

为了避免移动离得远的点, 我们选择一个光滑映射 $\lambda : \mathbf{R}^p \to \mathbf{R}$ 满足条件

$$\lambda(u) = \begin{cases} 1, & \text{当 } \|u\| \leqslant c/2 \text{ 时;} \\ 0, & \text{当 } \|u\| \geqslant c \text{ 时.} \end{cases}$$

现在, 同伦

$$F_t(x,u) = [1 - \lambda(u)t]F(x,u) + \lambda(u)tG(x,u)$$

将 $F = F_0$ 形变为映射 $F_1$, 使得: (1) $F_1$ 在区域 $\|u\| < c/2$ 中与 $G$ 重合; (2) 当 $\|u\| \geqslant c$ 时 $F_1$ 与 $F$ 重合; (3) $F_1$ 没有新的零点. 对最初的映射 $f$ 作一个相应的形变, 这就完成了引理 4 的证明.

**定理 B 的证明** 若 $f$ 与 $g$ 是光滑同伦的, 则引理 3 断定 Pontryagin 流形 $f^{-1}(y)$ 与 $g^{-1}(y)$ 是标架式协边的. 反之, 给定 $f^{-1}(y)$ 与 $g^{-1}(y)$ 之间的一个标架式协边 $(X, \mathfrak{m})$, 完全类似于定理 C 的证明, 可构造一个同伦

$$F : M \times [0,1] \longrightarrow S^p.$$

它的 Pontryagin 流形 $(F^{-1}(y), F^*\mathfrak{b})$ 正好等于 $(X, \mathfrak{m})$. 设 $F_t(x) = F(x,t)$, 注意映射 $F_0$ 与 $f$ 正好有着相同的 Pontryagin 流形. 因此根据引理 4, 有 $F_0 \sim f$. 类似地, $F_1 \sim g$. 因此 $f \sim g$. 这就完成了定理 B 的证明.

**注记** 定理 A, B 和 C 容易推广到使之对于有边流形 $M$ 适用. 根本的想法在于只考虑把边界变为某一基本点 $s_0$ 的映射. 这种映射

$$(M, \partial M) \longrightarrow (S^p, s_0)$$

的同伦类与余维数为 $p$ 的标架式子流形

$$N \subset M \text{ 的内部}$$

的协边类一一对应. 若 $p \geqslant \frac{1}{2}m + 1$, 则可对这个同伦类的集合给予一个交换群的构造, 并称之为第 $p$ 个同伦群 (cohomotopy group) $\pi^p(M, \partial M)$. $\pi^p(M, \partial M)$ 中的复合运算对应着 $M$ 内部中的无交的标架式子流形的并运算. (参见第 8 章, 问题 17.)

## Hopf 定理

作为例子, 设 $M$ 为一个 $m = p$ 维的连通的有向流形. 余维数为 $p$ 的标架式子流形恰好为一个有限点集, 且在每一点处有一指定的基. 令 $\operatorname{sgn}(x)$ 等于 $+1$ 或 $-1$, 按这个指定的基决定的定向是正确的定向还是错误的定向而定. 于是 $\sum \operatorname{sgn}(x)$ 显然等于相关的映射 $M \to S^m$ 的度. 但不难发现 0 维流形的标架式协边类完全由整数 $\sum \operatorname{sgn}(x)$ 决定. 于是证明了下列定理:

**Hopf 定理**　*若 $M$ 是连通的、有向的无边流形, 则两个映射 $M \to S^m$ 是光滑同伦的当且仅当它们有相同的度.*

另一方面, 设 $M$ 是不可定向的. 则给定 $TM_x$ 的一个基, 我们能够将 $x$ 围绕着 $M$ 在一条闭圈上滑行一周, 使得给定的基变成相反的定向. 用一个简单的推导便证得下列定理:

**定理**　*若 $M$ 是连通的, 但不可定向的流形, 则两个映射是同伦的当且仅当它们具有相同的模 2 度.*

标架式协边理论是由 Pontryagin 为了研究映射

$$S^m \longrightarrow S^p \quad (m > p)$$

的同伦类而引进的. 例如, 如果 $m = p + 1 \geqslant 4$, 则正好存在映射 $S^m \to S^p$ 的两个同伦类. Pontryagin 根据 $S^m$ 中标架式 1 维流形的分类证明了这个结果. 更难得的是, 他还指出在 $m = p + 2 \geqslant 4$ 的情形下也恰好存在两个同伦类. 这用到了标架式 2 维流形. 但是, 对于 $m - p > 2$, 用这种方法解决问题会碰到流形上的许多困难.

后来人们证实了用很不相同的, 并且更为代数化的方法[①]来列举同伦群是更为简易的. 尽管如此, Pontryagin 构造是一个双边的工具. 它不仅允许我们把流形的信息传入同伦理论, 还能使我们把同伦的任何信息传入流形理论. 在现代拓扑中的某些最深层次的工作就起源于这两种理论的交互作用, R. Thom 关于协边理论的工作便是一个重要的例子 (参见 [36], [21]).

① 例子请参见 S.T. Hu, *Homotopy Theory*.

# 第 8 章　练　　习

这里集中为读者列举一些问题:

**问题 1**　证明复合映射 $g \circ f$ 的度等于乘积 $(\deg g)(\deg f)$.

**问题 2**　证明每一个 $n$ 阶复多项式引出一个从 Gauss 球 $S^2$ 到自身的 $n$ 阶光滑映射.

**问题 3**　若从 $X$ 到 $S^p$ 的两个映射 $f$ 与 $g$ 满足对于所有的 $x$, $\| f(x) - g(x) \| < 2$, 证明 $f$ 同伦于 $g$. 且若 $f$ 与 $g$ 都是光滑的, 则这个同伦也是光滑的.

**问题 4**　若 $X$ 是紧致的, 证明每一个连续映射 $X \to S^p$ 均能用光滑映射一致地逼近. 若两个光滑映射 $X \to S^p$ 是连续同伦的, 证明它们还是光滑同伦的.

**问题 5**　若 $m < p$, 证明每一个映射 $M^m \to S^p$ 都同伦于常值映射.

**问题 6**　(Brouwer) 证明具有异于 $(-1)^{n+1}$ 的度的映射 $S^n \to S^n$ 必有一个不动点.

**问题 7**　证明度为奇数的任何一个映射 $S^n \to S^n$ 必将某一对对径点变为一对对径点.

**问题 8**　给定光滑流形 $M \subset \mathbf{R}^k$ 及 $N \subset \mathbf{R}^l$. 证明切空间 $T(M \times N)_{(x,y)}$ 等于 $TM_x \times TN_y$.

**问题 9**　光滑映射 $f : M \to N$ 的图 $\Gamma$ 定义为满足条件 $f(x) = y$ 的所有点 $(x, y) \in M \times N$ 的集合. 证明 $\Gamma$ 为一个光滑流形, 且切空间

$$T\Gamma_{(x,y)} \subset TM_x \times TN_y$$

等于线性映射 $\mathrm{d}f_x$ 的图.

**问题 10**　给定 $M \subset \mathbf{R}^k$, 证明切丛空间 (tangent bundle space)

$$TM = \{(x, v) \in M \times \mathbf{R}^k \mid v \in TM_x\}$$

也是一个光滑流形. 证明任何一个光滑映射 $f: M \to N$ 引出一个光滑映射

$$\mathrm{d}f: TM \longrightarrow TN,$$

其中

$$\mathrm{d}(\text{恒同}) = \text{恒同},$$
$$\mathrm{d}(g \circ f) = (\mathrm{d}g) \circ (\mathrm{d}f).$$

**问题 11**　同样证明: **法丛空间** (normal bundle space)

$$E = \{(x, v) \in M \times \mathbf{R}^k \mid v \perp TM_x\}$$

为一个光滑流形. 若 $M$ 是紧致且无边的, 证明从 $E$ 到 $\mathbf{R}^k$ 的对应

$$(x, v) \to x + v$$

将 $M \times 0$ 在 $E$ 中的 $\epsilon$ 邻域微分同胚地映射到 $M$ 在 $\mathbf{R}^k$ 中的 $\epsilon$ 邻域 $N_\epsilon$ 上. (参见第 7 章中的乘积邻域定理.)

**问题 12**　用 $r(x+v) = x$ 定义 $r: N_\epsilon \to M$. 证明 $r(x+v)$ 比 $M$ 的任何其他点更接近于 $x+v$. 用收缩 $r$ 证明与问题 4 类似的论断, 在其中将球 $S^p$ 换成流形 $M$.

**问题 13**　给定不相交的流形 $M, N \subset \mathbf{R}^{k+1}$, **环绕映射** (linking map)

$$\lambda: M \times N \longrightarrow S^k$$

定义为 $\lambda(x, y) = (x - y)/\|x - y\|$. 若 $M$ 与 $N$ 是紧致的、有向的, 且无边的, 其总维数为 $m + n = k$, 则 $\lambda$ 的度称为**环绕数** (linking number) $l(M, N)$. 证明

$$l(N, M) = (-1)^{(m+1)(n+1)} l(M, N).$$

如果 $M$ 为某一个与 $N$ 无交的有向流形 $X$ 的边, 证明 $l(M, N) = 0$. 对于在球 $S^{m+n+1}$ 中的无交流形定义**环绕数** (linking number).

**问题 14**　Hopf 不变量. 若 $y \neq z$ 都是映射 $f: S^{2p-1} \to S^p$ 的正则值, 于是流形 $f^{-1}(y)$, $f^{-1}(z)$ 能够像在第 5 章中那样给予定向, 因此环绕数 $l(f^{-1}(y), f^{-1}(z))$ 是有定义的.

a) 证明这一环绕数作为 $y$ 的函数是局部常值的;

b) 若 $y$ 与 $z$ 也是 $g$ 的正则值, 其中对于所有的 $x$,

$$\| f(x) - g(x) \| < \| y - z \| .$$

证明

$$l(f^{-1}(y), f^{-1}(z)) = l(g^{-1}(y), f^{-1}(z)) = l(g^{-1}(y), g^{-1}(z)).$$

c) 证明 $l(f^{-1}(y), f^{-1}(z))$ 仅依赖于 $f$ 的同伦类, 而不依赖于 $y$ 和 $z$ 的选择.

整数 $H(f) = l(f^{-1}(y), f^{-1}(z))$ 称为 $f$ 的 Hopf 不变量 (Hopf invariant) (见参考文献 [15]).

**问题 15**　若维数 $p$ 为奇数, 证明 $H(f) = 0$. 对于复合映射

$$S^{2p-1} \xrightarrow{f} S^p \xrightarrow{g} S^p$$

证明 $H(g \circ f)$ 等于 $H(f)$ 与 $g$ 的度的平方的乘积.

Hopf 纤维化 (Hopf fibration) $\pi : S^3 \to S^2$ 定义为

$$\pi(x_1, x_2.x_3, x_4) = h^{-1}((x_1 + \mathrm{i}x_2)/(x_3 + \mathrm{i}x_4)),$$

其中 $h$ 表示到复平面的球极平面投影. 证明 $H(\pi) = 1$.

**问题 16**　$M$ 的两个子流形 $N$ 与 $N'$ 称为横截相交的 (intersect tranversally), 如果对于每一个 $x \in N \cap N'$, 子空间 $TN_x$ 与 $TN'_x$ 共同生成 $TM_x$. (若 $n + n' < m$ , 这表示 $N \cap N' = \varnothing$.) 若 $N$ 为一个标架式子流形, 证明它能作一微小的形变使之与一给定的 $N'$ 横截相交. 证明得到的交为一个光滑流形.

**问题 17**　令 $\Pi^p(M)$ 表示所有 $M$ 中余维数为 $p$ 的标架式协边类的集合. 用横截相交运算定义一个对应

$$\Pi^p(M) \times \Pi^q(M) \longrightarrow \Pi^{p+q}(M)$$

若 $p \geqslant \frac{1}{2}m + 1$, 用无交并运算将 $\Pi^p(M)$ 作成一个交换群. (参见 p. 47.)

# 附录　1 维流形的分类

我们要证明在本书正文中假定为已知的下列结果, 还要给出关于高维流形分类问题的一个简短讨论.

**定理**　*任何一个光滑的连通的 1 维流形或者微分同胚于圆周 $S^1$ 或者微分同胚于实数的某一区间.*

(区间是 $\mathbf{R}$ 中多于一个点的连通子集, 它可以是有限的或无限的, 也可以是开的、闭的或半开的.)

因为任何一个区间都同胚①于 $[0,1]$, $(0,1]$ 或 $(0,1)$, 由此推出只有 4 个不同的 1 维流形.

证明要用到"弧长"的概念. 令 $I$ 表示一个区间.

**定义**　如果 $f$ 将 $I$ 微分同胚地映射到 $M$ 的一个开子集上②并且对于每一个 $s \in I$, "速度向量" $\mathrm{d}f_s(1) \in TM_{f(x)}$ 具有单位长, 则映射 $f: I \to M$ 称为一个*弧长式参数化* (parametrization by arc-length).

用一个直接的变量替换, 任何一个局部参数化 $I' \to M$ 都能变换成一个弧长式参数化.

**引理**　*令 $f: I \to M, g: J \to M$ 都是弧长式参数化. 则 $f(I) \cap g(J)$ 最多有两个分支. 若它只有一个分支, 那么 $f$ 能扩充为并 $f(I) \cup g(J)$ 的弧长式参数化. 若它有两个分支, 那么 $M$ 必定微分同胚于 $S^1$.*

**证明**　显然 $g^{-1} \circ f$ 将 $I$ 的某一相对开子集微分同胚地映射到 $J$ 的一个相对开子集上. 并且 $g^{-1} \circ f$ 的导数处处等于 $\pm 1$.

考虑由满足 $f(s) = g(t)$ 的所有 $(s, t)$ 组成的图形 $\Gamma \subset I \times J$. 则 $\Gamma$ 为由斜率为 $\pm 1$ 的线段作成的 $I \times J$ 的闭子集. 由于 $\Gamma$ 是闭的, 并且 $g^{-1} \circ f$ 局部地为一个微分同胚, 这些线段不能在 $I \times J$ 的内部有端点, 而是必定能扩展到边上. 由于 $g^{-1} \circ f$ 是一一且单值的, 在矩形 $I \times J$ 的四条边的每一条边上, 这些线段中最多只有一条能够取端点. 因此 $\Gamma$ 最多有两个分支. (见图 19.) 而且当有两个分支时, 两者必有同样的

---

① 例如, 用形如

$$f(t) = a \tanh(t) + b$$

的微分同胚.

② 于是仅当 $M$ 有边点时, $I$ 才能有边点.

斜率.

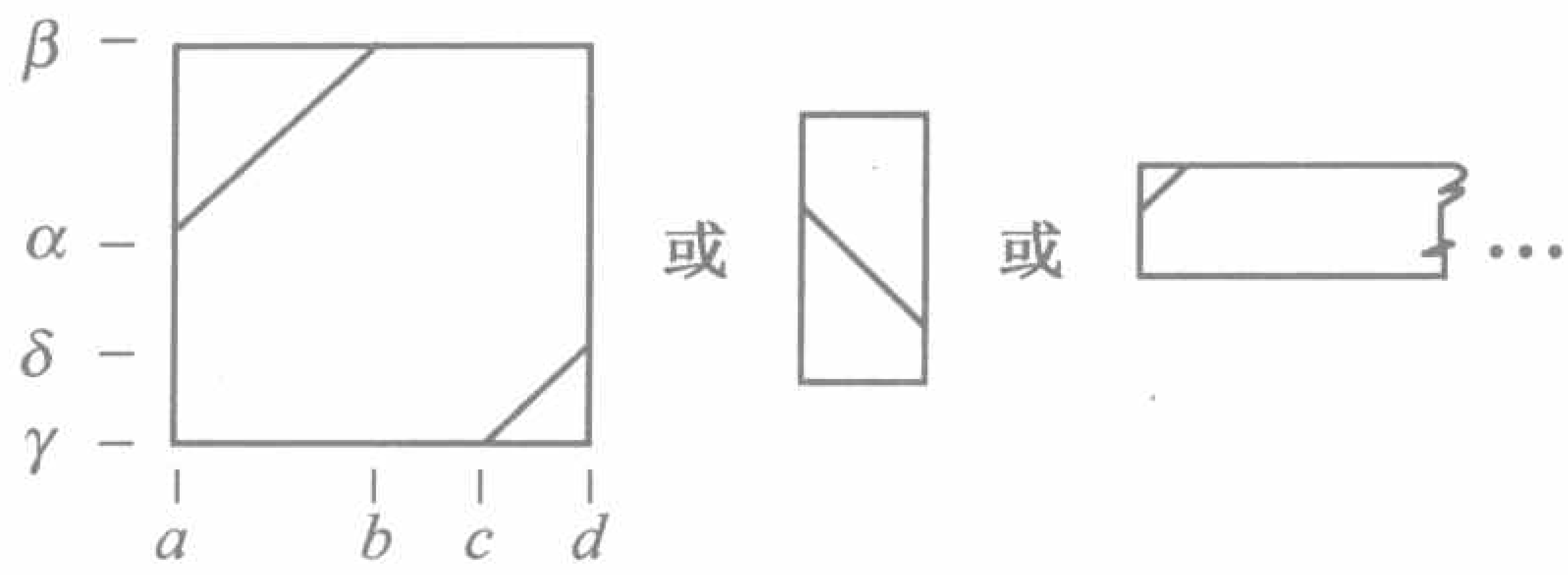


图 19 Γ 的 3 种可能性

若 $\Gamma$ 是连通的, 则 $g^{-1} \circ f$ 扩充为一个线性映射 $L: \mathbf{R} \to \mathbf{R}$. 将 $f$ 及 $g \circ L$ 粘接起来即产生所求的扩充

$$F: I \cup L^{-1}(J) \longrightarrow f(I) \cup g(J).$$

若 $\Gamma$ 有两个分支, 设其斜率为 $+1$, 它们必能被安排得像在图 19 的左边矩形中一样, 必要时, 改造一下区间 $J = (\gamma, \beta)$, 可以假定 $\gamma = c$, $\delta = d$, 于是

$$a < b \leqslant c < d \leqslant \alpha < \beta.$$

现设 $\theta = 2\pi t/(\alpha - a)$, 所求的微分同胚

$$h: S^1 \longrightarrow M$$

可用下面的公式来定义:

$$h(\cos\theta, \sin\theta) = \begin{cases} f(t), & \text{当 } a < t < d \text{ 时;} \\ g(t), & \text{当 } c < t < \beta \text{ 时.} \end{cases}$$

象集 $h(S^1)$ 在 $M$ 中是紧致的开集, 故必为整个流形 $M$. 引理得证.

**分类定理的证明** 任何一个弧长式参数化都能扩充为最大的弧长式参数化

$$f: I \longrightarrow M$$

其中“最大”的意思是: $f$ 作为弧长式参数化而言, 不能再扩充到更大的区间上: 这只要将 $f$ 先尽可能地向左扩充, 然后再尽可能地向右扩充.

若 $M$ 不微分同胚于 $S^1$, 我们要证明 $f$ 是满的, 因此是一个微分同胚. 因为如果开集 $f(I)$ 不是整个 $M$, 那么在 $M - f(I)$ 中有 $f(I)$ 的一个极限点 $x$. 将 $x$ 的一个邻域弧长式参数化, 并且应用引理, 可以看到 $f$ 能够扩充到一个比较大的区间上. 这与 $f$ 是最大的假定相矛盾. 从而完成证明.

**注记**　高维流形的分类问题令人难以回答. 对于 2 维流形, 完整的说明是由 Kerekjarto [17] 给出的. 3 维流形的研究是当前的研究专题 (见 Papakyriakopoulos [26]). 对于维数 $\geqslant 4$ 的紧致流形, 分类问题实际上是不可解决的[①]. 但对于高维单连通的流形, 近年来有许多进展, 例证见 Smale [31] 和 Wall[37].

① 见 Markov [19].

# 参考文献

下面的清单[①]列出了原始资料和推荐给读者的参考书. 对于希望探究微分拓扑学的读者, 我们推荐 Milnor [22]、Munkres [25] 及 Pontryagin [28]. 综述性文章 [23] 和 [32] 也是有用的. 若想了解密切相关领域中的背景知识, 我们推荐 Hilton and Wylie [11]、Hu [16]、Lang [18]、de Rham [29]、Steenrod [34] 以及 Sternberg [35].



① 为避免重复, 此处没有列出, 见后面英文部分的参考文献. —— 编者注

# 索　引

### 符号索引



### 英文首字



### 中文首字

# §1. SMOOTH MANIFOLDS AND SMOOTH MAPS

FIRST let us explain some of our terms. $R^k$ denotes the $k$-dimensional euclidean space; thus a point $x \in R^k$ is an $k$-tuple $x = (x_1, \cdots, x_k)$ of real numbers.

Let $U \subset R^k$ and $V \subset R^l$ be open sets. A mapping $f$ from $U$ to $V$ (written $f : U \to V$) is called *smooth* if all of the partial derivatives $\partial^n f/\partial x_{i_1} \cdots \partial x_{i_n}$ exist and are continuous.

More generally let $X \subset R^k$ and $Y \subset R^l$ be arbitrary subsets of euclidean spaces. A map $f : X \to Y$ is called *smooth* if for each $x \in X$ there exist an open set $U \subset R^k$ containing $x$ and a smooth mapping $F : U \to R^l$ that coincides with $f$ throughout $U \cap X$.

If $f : X \to Y$ and $g : Y \to Z$ are smooth, note that the composition $g \circ f : X \to Z$ is also smooth. The identity map of any set $X$ is automatically smooth.

DEFINITION. A map $f : X \to Y$ is called a *diffeomorphism* if $f$ carries $X$ homeomorphically onto $Y$ and if both $f$ and $f^{-1}$ are smooth.

We can now indicate roughly what *differential topology* is about by saying that it studies those properties of a set $X \subset R^k$ which are invariant under diffeomorphism.

We do not, however, want to look at completely arbitrary sets $X$. The following definition singles out a particularly attractive and useful class.

DEFINITION. A subset $M \subset R^k$ is called a *smooth manifold* of *dimension* $m$ if each $x \in M$ has a neighborhood $W \cap M$ that is diffeomorphic to an open subset $U$ of the euclidean space $R^m$.

Any particular diffeomorphism $g : U \to W \cap M$ is called a *parametrization* of the region $W \cap M$. (The inverse diffeomorphism $W \cap M \to U$ is called a system of *coordinates* on $W \cap M$.)

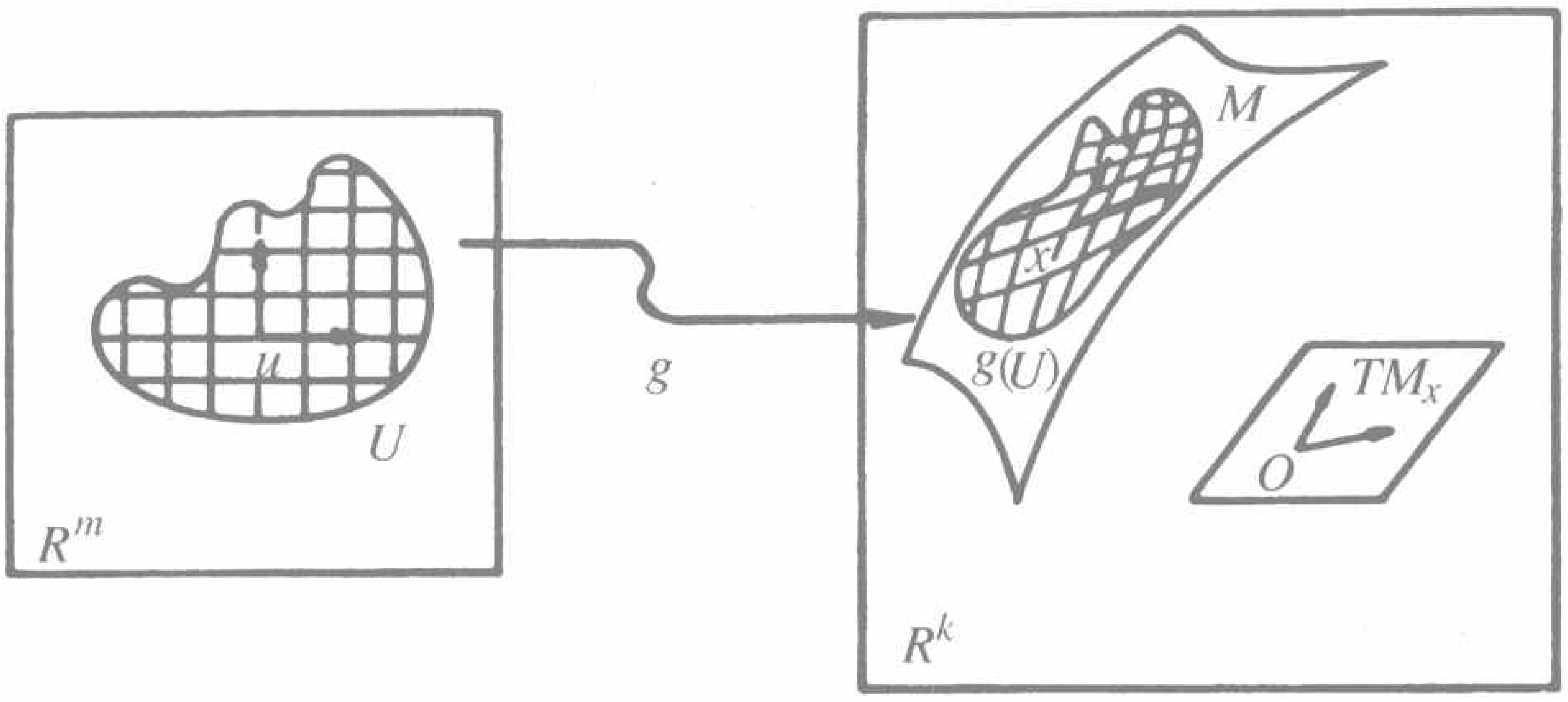


*Figure 1. Parametrization of a region in M*

Sometimes we will need to look at manifolds of dimension zero. By definition, $M$ is a manifold of dimension zero if each $x \in M$ has a neighborhood $W \cap M$ consisting of $x$ alone.

EXAMPLES. The unit sphere $S^2$, consisting of all $(x, y, z) \in R^3$ with $x^2 + y^2 + z^2 = 1$ is a smooth manifold of dimension 2. In fact the diffeomorphism

$$(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2}),$$

for $x^2 + y^2 < 1$, parametrizes the region $z > 0$ of $S^2$. By interchanging the roles of $x$, $y$, $z$, and changing the signs of the variables, we obtain similar parametrizations of the regions $x > 0$, $y > 0$, $x < 0$, $y < 0$, and $z < 0$. Since these cover $S^2$, it follows that $S^2$ is a smooth manifold.

More generally the sphere $S^{n-1} \subset R^n$ consisting of all $(x_1, \cdots, x_n)$ with $\sum x_i^2 = 1$ is a smooth manifold of dimension $n - 1$. For example $S^0 \subset R^1$ is a manifold consisting of just two points.

A somewhat wilder example of a smooth manifold is given by the set of all $(x, y) \in R^2$ with $x \neq 0$ and $y = \sin(1/x)$.

## TANGENT SPACES AND DERIVATIVES

To define the notion of *derivative* $df_x$ for a smooth map $f : M \to N$ of smooth manifolds, we first associate with each $x \in M \subset R^k$ a linear subspace $TM_x \subset R^k$ of dimension $m$ called the *tangent space* of $M$ at $x$. Then $df_x$ will be a linear mapping from $TM_x$ to $TN_y$, where $y = f(x)$. Elements of the vector space $TM_x$ are called *tangent vectors* to $M$ at $x$.

Intuitively one thinks of the $m$-dimensional hyperplane in $R^k$ which best approximates $M$ near $x$; then $TM_x$ is the hyperplane through the

origin that is parallel to this. (Compare Figures 1 and 2.) Similarly one thinks of the nonhomogeneous linear mapping from the tangent hyperplane at $x$ to the tangent hyperplane at $y$ which best approximates $f$. Translating both hyperplanes to the origin, one obtains $df_x$.

Before giving the actual definition, we must study the special case of mappings between open sets. For any open set $U \subset R^k$ the *tangent space* $TU_x$ is defined to be the entire vector space $R^k$. For any smooth map $f\colon U \to V$ the *derivative*

$$df_x : R^k \to R^l$$

is defined by the formula

$$df_x(h) = \lim_{t \to 0} (f(x + th) - f(x))/t$$

for $x \in U$, $h \in R^k$. Clearly $df_x(h)$ is a linear function of $h$. (In fact $df_x$ is just that linear mapping which corresponds to the $l \times k$ matrix $(\partial f_i/\partial x_j)_x$ of first partial derivatives, evaluated at $x$.)

Here are two fundamental properties of the derivative operation:

1 (Chain rule). *If $f : U \to V$ and $g : V \to W$ are smooth maps, with $f(x) = y$, then*

$$d(g \circ f)_x = dg_y \circ df_x.$$

In other words, to every commutative triangle

$$\begin{tikzcd} & V \arrow[dr, "g"] & \\ U \arrow[ur, "f"] \arrow[rr, "g \circ f"'] & & W \end{tikzcd}$$

of smooth maps between open subsets of $R^k$, $R^l$, $R^m$ there corresponds a commutative triangle of linear maps

$$\begin{tikzcd} & R^l \arrow[dr, "dg_y"] & \\ R^k \arrow[ur, "df_x"] \arrow[rr, "d(g \circ f)_x"'] & & R^m. \end{tikzcd}$$

2. *If $I$ is the identity map of $U$, then $dI_x$ is the identity map of $R^k$. More generally, if $U \subset U'$ are open sets and*

$$i : U \to U'$$

*is the inclusion map, then again $di_x$ is the identity map of $R^k$.*

Note also:

3. *If $L : R^k \to R^l$ is a linear mapping, then $dL_x = L$.*

As a simple application of the two properties one has the following:

ASSERTION. *If $f$ is a diffeomorphism between open sets $U \subset R^k$ and $V \subset R^l$, then $k$ must equal $l$, and the linear mapping*

$$df_x : R^k \to R^l$$

*must be nonsingular.*

PROOF. The composition $f^{-1} \circ f$ is the identity map of $U$; hence $d(f^{-1})_y \circ df_x$ is the identity map of $R^k$. Similarly $df_x \circ d(f^{-1})_y$ is the identity map of $R^l$. Thus $df_x$ has a two-sided inverse, and it follows that $k = l$.

A partial converse to this assertion is valid. Let $f : U \to R^k$ be a smooth map, with $U$ open in $R^k$.

**Inverse Function Theorem.** *If the derivative $df_x : R^k \to R^k$ is nonsingular, then $f$ maps any sufficiently small open set $U'$ about $x$ diffeomorphically onto an open set $f(U')$.*

(See Apostol [2, p. 144] or Dieudonne [7, p. 268].)

Note that $f$ may not be one-one in the large, even if every $df_x$ is nonsingular. (An instructive example is provided by the exponential mapping of the complex plane into itself.)

Now let us define the *tangent space* $TM_x$ for an arbitrary smooth manifold $M \subset R^k$. Choose a parametrization

$$g : U \to M \subset R^k$$

of a neighborhood $g(U)$ of $x$ in $M$, with $g(u) = x$. Here $U$ is an open subset of $R^m$. Think of $g$ as a mapping from $U$ to $R^k$, so that the derivative

$$dg_u : R^m \to R^k$$

is defined. *Set $TM_x$ equal to the image $dg_u(R^m)$ of $dg_u$.* (Compare Figure 1.)

We must prove that this construction does not depend on the particular choice of parametrization $g$. Let $h : V \to M \subset R^k$ be another parametrization of a neighborhood $h(V)$ of $x$ in $M$, and let $v = h^{-1}(x)$. Then $h^{-1} \circ g$ maps some neighborhood $U_1$ of $u$ diffeomorphically onto a neighborhood $V_1$ of $v$. The commutative diagram of smooth maps

between open sets

$$\begin{tikzcd} & R^k & \\ U_1 \arrow[ur, "g"] \arrow[rr, "h^{-1} \circ g"'] & & V_1 \arrow[ul, "h"'] \end{tikzcd}$$

gives rise to a commutative diagram of linear maps

$$\begin{tikzcd} & R^k & \\ R^m \arrow[ur, "dg_u"] \arrow[rr, "\cong", "d(h^{-1} \circ g)_u"'] & & R^m, \arrow[ul, "dh_v"'] \end{tikzcd}$$

and it follows immediately that

$$\text{Image } (dg_u) = \text{Image } (dh_v).$$

Thus $TM_x$ is well defined.

PROOF THAT $TM_x$ IS AN $m$-DIMENSIONAL VECTOR SPACE. Since

$$g^{-1} : g(U) \to U$$

is a smooth mapping, we can choose an open set $W$ containing $x$ and a smooth map $F : W \to R^m$ that coincides with $g^{-1}$ on $W \cap g(U)$. Setting $U_0 = g^{-1}(W \cap g(U))$, we have the commutative diagram

$$\begin{tikzcd} & W \arrow[dr, "F"] & \\ U_0 \arrow[ur, "g"] \arrow[rr, "\text{inclusion}"] & & R^m \end{tikzcd}$$

and therefore

$$\begin{tikzcd} & R^k \arrow[dr, "dF_x"] & \\ R^m \arrow[ur, "dg_u"] \arrow[rr, "\text{identity}"] & & R^m \end{tikzcd}$$

This diagram clearly implies that $dg_u$ has rank $m$, and hence that its image $TM_x$ has dimension $m$.

Now consider two smooth manifolds, $M \subset R^k$ and $N \subset R^l$, and a

smooth map

$$f : M \to N$$

with $f(x) = y$. The *derivative*

$$df_x : TM_x \to TN_y$$

is defined as follows. Since $f$ is smooth there exist an open set $W$ containing $x$ and a smooth map

$$F : W \to R^l$$

that coincides with $f$ on $W \cap M$. *Define $df_x(v)$ to be equal to $dF_x(v)$ for all $v \in TM_x$.*

To justify this definition we must prove that $dF_x(v)$ belongs to $TN_y$ and that it does not depend on the particular choice of $F$.

Choose parametrizations

$$g : U \to M \subset R^k \quad \text{and} \quad h : V \to N \subset R^l$$

for neighborhoods $g(U)$ of $x$ and $h(V)$ of $y$. Replacing $U$ by a smaller set if necessary, we may assume that $g(U) \subset W$ and that $f$ maps $g(U)$ into $h(V)$. It follows that

$$h^{-1} \circ f \circ g : U \to V$$

is a well-defined smooth mapping.

Consider the commutative diagram

$$\begin{tikzcd} W \arrow[r, "F"] & R^l \\ U \arrow[u, "g"] \arrow[r, "h^{-1} \circ f \circ g"] & V \arrow[u, "h"'] \end{tikzcd}$$

of smooth mappings between open sets. Taking derivatives, we obtain a commutative diagram of linear mappings

$$\begin{tikzcd} R^k \arrow[r, "dF_x"] & R^l \\ R^m \arrow[u, "dg_u"] \arrow[r, "d(h^{-1} \circ f \circ g)_u"] & R^n \arrow[u, "dh_v"'] \end{tikzcd},$$

where $u = g^{-1}(x)$, $v = h^{-1}(y)$.

It follows immediately that $dF_x$ carries $TM_x = \text{Image}\ (dg_u)$ into $TN_y = \text{Image}\ (dh_v)$. Furthermore the resulting map $df_x$ does not depend on the particular choice of $F$, for we can obtain the same linear

transformation by going around the bottom of the diagram. That is:

$$df_x = dh_v \circ d(h^{-1} \circ f \circ g)_u \circ (dg_u)^{-1}.$$

This completes the proof that

$$df_x : TM_x \to TN_y$$

is a well-defined linear mapping.

As before, the derivative operation has two fundamental properties:

1. (Chain rule). *If $f : M \to N$ and $g : N \to P$ are smooth, with $f(x) = y$, then*

$$d(g \circ f)_x = dg_y \circ df_x.$$

2. *If $I$ is the identity map of $M$, then $dI_x$ is the identity map of $TM_x$. More generally, if $M \subset N$ with inclusion map $i$, then $TM_x \subset TN_x$ with inclusion map $di_x$.* (Compare Figure 2.)

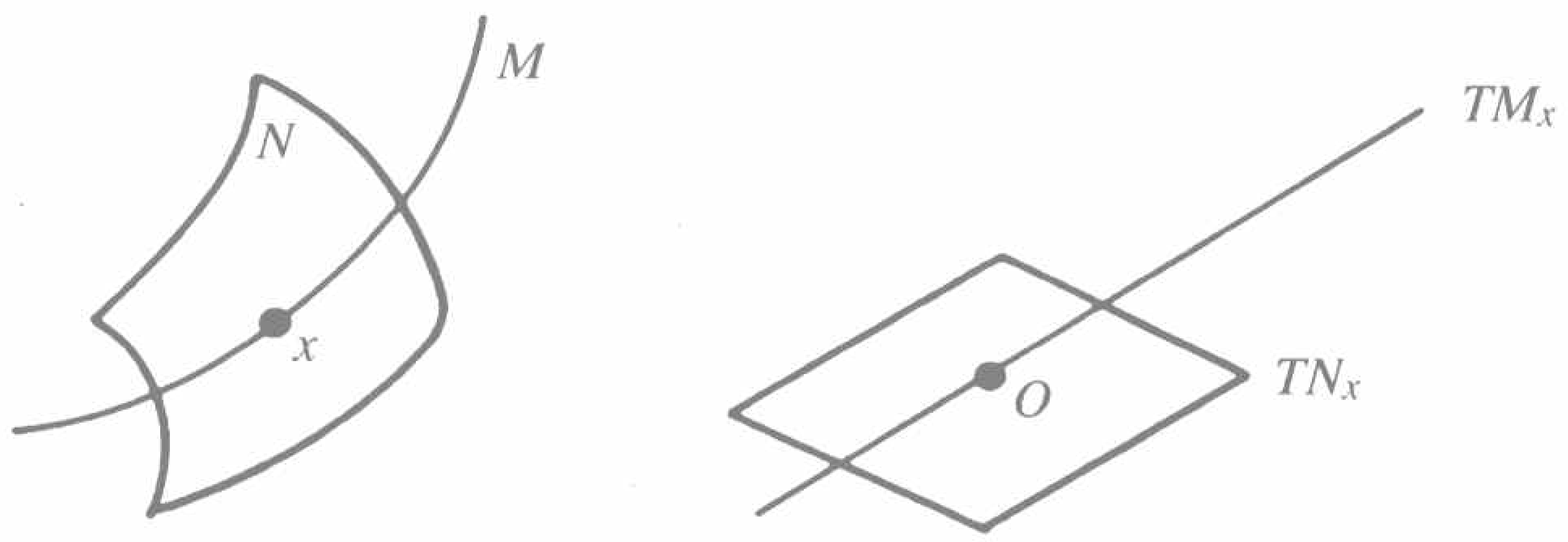


*Figure 2. The tangent space of a submanifold*

The proofs are straightforward.

As before, these two properties lead to the following:

Assertion. *If $f : M \to N$ is a diffeomorphism, then $df_x : TM_x \to TN_y$ is an isomorphism of vector spaces. In particular the dimension of $M$ must be equal to the dimension of $N$.*

## REGULAR VALUES

Let $f : M \to N$ be a smooth map between manifolds of the same dimension.* We say that $x \in M$ is a *regular point* of $f$ if the derivative

* This restriction will be removed in §2.

$df_x$ is nonsingular. In this case it follows from the inverse function theorem that $f$ maps a neighborhood of $x$ in $M$ diffeomorphically onto an open set in $N$. The point $y \in N$ is called a *regular value* if $f^{-1}(y)$ contains only regular points.

If $df_x$ is singular, then $x$ is called a *critical point* of $f$, and the image $f(x)$ is called a *critical value*. Thus each $y \in N$ is either a critical value or a regular value according as $f^{-1}(y)$ does or does not contain a critical point.

*Observe that if $M$ is compact and $y \in N$ is a regular value, then $f^{-1}(y)$ is a finite set (possibly empty).* For $f^{-1}(y)$ is in any case compact, being a closed subset of the compact space $M$; and $f^{-1}(y)$ is discrete, since $f$ is one-one in a neighborhood of each $x \in f^{-1}(y)$.

For a smooth $f : M \to N$, with $M$ compact, and a regular value $y \in N$, *we define $\#f^{-1}(y)$ to be the number of points in $f^{-1}(y)$.* The first observation to be made about $\#f^{-1}(y)$ is that it is locally constant as a function of $y$ (where $y$ ranges only through regular values!). I.e., *there is a neighborhood $V \subset N$ of $y$ such that $\#f^{-1}(y') = \#f^{-1}(y)$ for any $y' \in V$.* [Let $x_1, \cdots, x_k$ be the points of $f^{-1}(y)$, and choose pairwise disjoint neighborhoods $U_1, \cdots, U_k$ of these which are mapped diffeomorphically onto neighborhoods $V_1, \cdots, V_k$ in $N$. We may then take

$$V = V_1 \cap V_2 \cap \cdots \cap V_k - f(M - U_1 - \cdots - U_k).]$$

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## THE FUNDAMENTAL THEOREM OF ALGEBRA

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As an application of these notions, we prove the fundamental theorem of algebra: *every nonconstant complex polynomial $P(z)$ must have a zero.*

For the proof it is first necessary to pass from the plane of complex numbers to a compact manifold. Consider the unit sphere $S^2 \subset R^3$ and the stereographic projection

$$h_+ : S^2 - \{(0, 0, 1)\} \to R^2 \times 0 \subset R^3$$

from the "north pole" $(0, 0, 1)$ of $S^2$. (See Figure 3.) We will identify $R^2 \times 0$ with the plane of complex numbers. The polynomial map $P$ from $R^2 \times 0$ to itself corresponds to a map $f$ from $S^2$ to itself; where

$$f(x) = h_+^{-1} P h_+(x) \quad \text{for} \quad x \neq (0, 0, 1)$$

$$f(0, 0, 1) = (0, 0, 1).$$

It is well known that this resulting map $f$ is smooth, even in a neighbor-

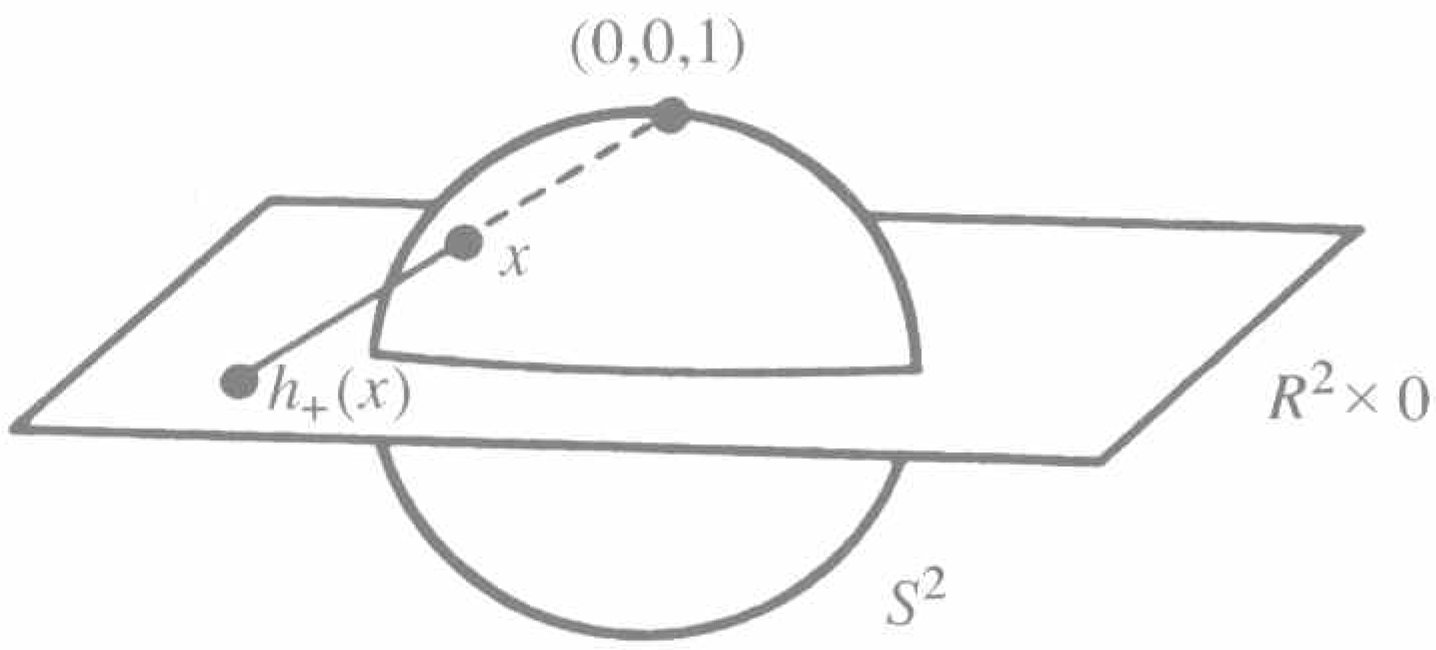



*Figure 3. Stereographic projection*

hood of the north pole. To see this we introduce the stereographic projection $h_-$ from the south pole $(0, 0, -1)$ and set

$$Q(z) = h_- f h_-^{-1}(z).$$

Note, by elementary geometry, that

$$h_+ h_-^{-1}(z) = z/|z|^2 = 1/\bar{z}.$$

Now if $P(z) = a_0 z^n + a_1 z^{n-1} + \cdots + a_n$, with $a_0 \neq 0$, then a short computation shows that

$$Q(z) = z^n/(\bar{a}_0 + \bar{a}_1 z + \cdots + \bar{a}_n z^n).$$

Thus $Q$ is smooth in a neighborhood of 0, and it follows that $f = h_-^{-1} Q h_-$ is smooth in a neighborhood of $(0, 0, 1)$.

Next observe that $f$ has only a finite number of critical points; for $P$ fails to be a local diffeomorphism only at the zeros of the derivative polynomial $P'(z) = \sum a_{n-j}\, j z^{j-1}$, and there are only finitely many zeros since $P'$ is not identically zero. The set of regular values of $f$, being a sphere with finitely many points removed, is therefore connected. Hence the locally constant function $\#f^{-1}(y)$ must actually be constant on this set. Since $\#f^{-1}(y)$ can't be zero everywhere, we conclude that it is zero nowhere. Thus $f$ is an onto mapping, and the polynomial $P$ must have a zero.

# §2. THE THEOREM OF SARD AND BROWN

IN GENERAL, it is too much to hope that the set of critical values of a smooth map be finite. But this set will be "small," in the sense indicated by the next theorem, which was proved by A. Sard in 1942 following earlier work by A. P. Morse. (References [30], [24].)

**Theorem.** *Let $f : U \to R^n$ be a smooth map, defined on an open set $U \subset R^m$, and let*

$$C = \{x \in U \mid \operatorname{rank} df_x < n\}.$$

*Then the image $f(C) \subset R^n$ has Lebesgue measure zero.**

Since a set of measure zero cannot contain any nonvacuous open set, it follows that the complement $R^n - f(C)$ must be everywhere dense† in $R^n$.

The proof will be given in §3. It is essential for the proof that $f$ should have many derivatives. (Compare Whitney [38].)

We will be mainly interested in the case $m \geq n$. If $m < n$, then clearly $C = U$; hence the theorem says simply that $f(U)$ has measure zero.

More generally consider a smooth map $f : M \to N$, from a manifold of dimension $m$ to a manifold of dimension $n$. Let $C$ be the set of all $x \in M$ such that

$$df_x : TM_x \to TN_{f(x)}$$

* In other words, given any $\epsilon > 0$, it is possible to cover $f(C)$ by a sequence of cubes in $R^n$ having total $n$-dimensional volume less than $\epsilon$.

† Proved by Arthur B. Brown in 1935. This result was rediscovered by Dubovickiĭ in 1953 and by Thom in 1954. (References [5], [8], [36].)

has rank less than $n$ (i.e. is not onto). Then $C$ will be called the set of *critical points*, $f(C)$ the set of *critical values*, and the complement $N - f(C)$ the set of *regular values* of $f$. (This agrees with our previous definitions in the case $m = n$.) Since $M$ can be covered by a countable collection of neighborhoods each diffeomorphic to an open subset of $R^m$, we have:

**Corollary** (A. B. Brown). *The set of regular values of a smooth map $f : M \to N$ is everywhere dense in $N$.*

In order to exploit this corollary we will need the following:

**Lemma 1.** *If $f : M \to N$ is a smooth map between manifolds of dimension $m \geq n$, and if $y \in N$ is a regular value, then the set $f^{-1}(y) \subset M$ is a smooth manifold of dimension $m - n$.*

PROOF. Let $x \in f^{-1}(y)$. Since $y$ is a regular value, the derivative $df_x$ must map $TM_x$ onto $TN_y$. The null space $\mathfrak{N} \subset TM_x$ of $df_x$ will therefore be an $(m - n)$-dimensional vector space.

If $M \subset R^k$, choose a linear map $L : R^k \to R^{m-n}$ that is nonsingular on this subspace $\mathfrak{N} \subset TM_x \subset R^k$. Now define

$$F : M \to N \times R^{m-n}$$

by $F(\xi) = (f(\xi), L(\xi))$. The derivative $dF_x$ is clearly given by the formula

$$dF_x(v) = (df_x(v), L(v)).$$

Thus $dF_x$ is nonsingular. Hence $F$ maps some neighborhood $U$ of $x$ diffeomorphically onto a neighborhood $V$ of $(y, L(x))$. Note that $f^{-1}(y)$ corresponds, under $F$, to the hyperplane $y \times R^{m-n}$. In fact $F$ maps $f^{-1}(y) \cap U$ diffeomorphically onto $(y \times R^{m-n}) \cap V$. This proves that $f^{-1}(y)$ is a smooth manifold of dimension $m - n$.

As an example we can give an easy proof that the unit sphere $S^{m-1}$ is a smooth manifold. Consider the function $f : R^m \to R$ defined by

$$f(x) = x_1^2 + x_2^2 + \cdots + x_m^2.$$

Any $y \neq 0$ is a regular value, and the smooth manifold $f^{-1}(1)$ is the unit sphere.

If $M'$ is a manifold which is contained in $M$, it has already been noted that $TM'_x$ is a subspace of $TM_x$ for $x \in M'$. The orthogonal complement of $TM'_x$ in $TM_x$ is then a vector space of dimension $m - m'$ called *the space of normal vectors to $M'$ in $M$ at $x$.*

In particular let $M' = f^{-1}(y)$ for a regular value $y$ of $f : M \to N$.

**Lemma 2.** *The null space of $df_x : TM_x \to TN_y$ is precisely equal to the tangent space $TM'_x \subset TM_x$ of the submanifold $M' = f^{-1}(y)$. Hence $df_x$ maps the orthogonal complement of $TM'_x$ isomorphically onto $TN_y$.*

PROOF. From the diagram

$$\begin{array}{ccc} M' & \xrightarrow{\ i\ } & M \\ \downarrow & & \downarrow f \\ y & \longrightarrow & N \end{array}$$

we see that $df_x$ maps the subspace $TM'_x \subset TM_x$ to zero. Counting dimensions we see that $df_x$ *maps the space of normal vectors to* $M'$ *isomorphically onto* $TN_y$.

## MANIFOLDS WITH BOUNDARY

The lemmas above can be sharpened so as to apply to a map defined on a smooth "manifold with boundary." Consider first the closed half-space

$$H^m = \{(x_1, \cdots, x_m) \in R^m \mid x_m \geq 0\}.$$

The *boundary* $\partial H^m$ is defined to be the hyperplane $R^{m-1} \times 0 \subset R^m$.

DEFINITION. A subset $X \subset R^k$ is called a *smooth m-manifold with boundary* if each $x \in X$ has a neighborhood $U \cap X$ diffeomorphic to an open subset $V \cap H^m$ of $H^m$. The *boundary* $\partial X$ is the set of all points in $X$ which correspond to points of $\partial H^m$ under such a diffeomorphism.

It is not hard to show that $\partial X$ is a well-defined smooth manifold of dimension $m - 1$. The *interior* $X - \partial X$ is a smooth manifold of dimension $m$.

The tangent space $TX_x$ is defined just as in §1, so that $TX_x$ is a full $m$-dimensional vector space, even if $x$ is a boundary point.

Here is one method for generating examples. Let $M$ be a manifold without boundary and let $g : M \to R$ have 0 as regular value.

**Lemma 3.** *The set of $x$ in $M$ with $g(x) \geq 0$ is a smooth manifold, with boundary equal to $g^{-1}(0)$.*

The proof is just like the proof of Lemma 1.

Example. The *unit disk* $D^m$, consisting of all $x \in R^m$ with

$$1 - \sum x_i^2 \geq 0,$$

is a smooth manifold, with boundary equal to $S^{m-1}$.

Now consider a smooth map $f : X \to N$ from an $m$-manifold with boundary to an $n$-manifold, where $m > n$.

**Lemma 4.** *If $y \in N$ is a regular value, both for $f$ and for the restriction $f \mid \partial X$, then $f^{-1}(y) \subset X$ is a smooth $(m - n)$-manifold with boundary. Furthermore the boundary $\partial(f^{-1}(y))$ is precisely equal to the intersection of $f^{-1}(y)$ with $\partial X$.*

Proof. Since we have to prove a local property, it suffices to consider the special case of a map $f : H^m \to R^n$, with regular value $y \in R^n$. Let $\bar{x} \in f^{-1}(y)$. If $\bar{x}$ is an interior point, then as before $f^{-1}(y)$ is a smooth manifold in the neighborhood of $\bar{x}$.

Suppose that $\bar{x}$ is a boundary point. Choose a smooth map $g : U \to R^n$ that is defined throughout a neighborhood of $\bar{x}$ in $R^m$ and coincides with $f$ on $U \cap H^m$. Replacing $U$ by a smaller neighborhood if necessary, we may assume that $g$ has no critical points. Hence $g^{-1}(y)$ is a smooth manifold of dimension $m - n$.

Let $\pi : g^{-1}(y) \to R$ denote the coordinate projection,

$$\pi(x_1, \cdots, x_m) = x_m.$$

We claim that $\pi$ has 0 as a regular value. For the tangent space of $g^{-1}(y)$ at a point $x \in \pi^{-1}(0)$ is equal to the null space of

$$dg_x = df_x : R^m \to R^n;$$

but the hypothesis that $f \mid \partial H^m$ is regular at $x$ guarantees that this null space cannot be completely contained in $R^{m-1} \times 0$.

Therefore the set $g^{-1}(y) \cap H^m = f^{-1}(y) \cap U$, consisting of all $x \in g^{-1}(y)$ with $\pi(x) \geq 0$, is a smooth manifold, by Lemma 3; with boundary equal to $\pi^{-1}(0)$. This completes the proof.

## THE BROUWER FIXED POINT THEOREM

We now apply this result to prove the key lemma leading to the classical Brouwer fixed point theorem. Let $X$ be a compact manifold with boundary.

**Lemma 5.** *There is no smooth map $f : X \to \partial X$ that leaves $\partial X$ pointwise fixed.*

Proof (following M. Hirsch). Suppose there were such a map $f$. Let $y \in \partial X$ be a regular value for $f$. Since $y$ is certainly a regular value for the identity map $f \mid \partial X$ also, it follows that $f^{-1}(y)$ is a smooth 1-manifold, with boundary consisting of the single point

$$f^{-1}(y) \cap \partial X = \{y\}.$$

But $f^{-1}(y)$ is also compact, and the only compact 1-manifolds are finite disjoint unions of circles and segments,* so that $\partial f^{-1}(y)$ must consist of an even number of points. This contradiction establishes the lemma.

In particular the unit disk

$$D^n = \{x \in R^n \mid x_1^2 + \cdots + x_n^2 \leq 1\}$$

is a compact manifold bounded by the unit sphere $S^{n-1}$. Hence as a special case we have proved that *the identity map of $S^{n-1}$ cannot be extended to a smooth map $D^n \to S^{n-1}$.*

**Lemma 6.** *Any smooth map $g : D^n \to D^n$ has a fixed point* (i.e. a point $x \in D^n$ with $g(x) = x$).

Proof. Suppose $g$ has no fixed point. For $x \in D^n$, let $f(x) \in S^{n-1}$ be the point nearer $x$ than $g(x)$ on the line through $x$ and $g(x)$. (See Figure 4.) Then $f : D^n \to S^{n-1}$ is a smooth map with $f(x) = x$ for $x \in S^{n-1}$, which is impossible by Lemma 5. (To see that $f$ is smooth we make the following explicit computation: $f(x) = x + tu$, where

$$u = \frac{x - g(x)}{||x - g(x)||}, \qquad t = -x\cdot u + \sqrt{1 - x\cdot x + (x\cdot u)^2},$$

the expression under the square root sign being strictly positive. Here and subsequently $||x||$ denotes the euclidean length $\sqrt{x_1^2 + \cdots + x_n^2}$.)

**Brouwer Fixed Point Theorem.** *Any continuous function $G : D^n \to D^n$ has a fixed point.*

Proof. We reduce this theorem to the lemma by approximating $G$ by a smooth mapping. Given $\epsilon > 0$, according to the Weierstrass approximation theorem,† there is a polynomial function $P_1 : R^n \to R^n$ with $||P_1(x) - G(x)|| < \epsilon$ for $x \in D^n$. However, $P_1$ may send points

* A proof is given in the Appendix.

† See for example Dieudonne [7, p. 133].

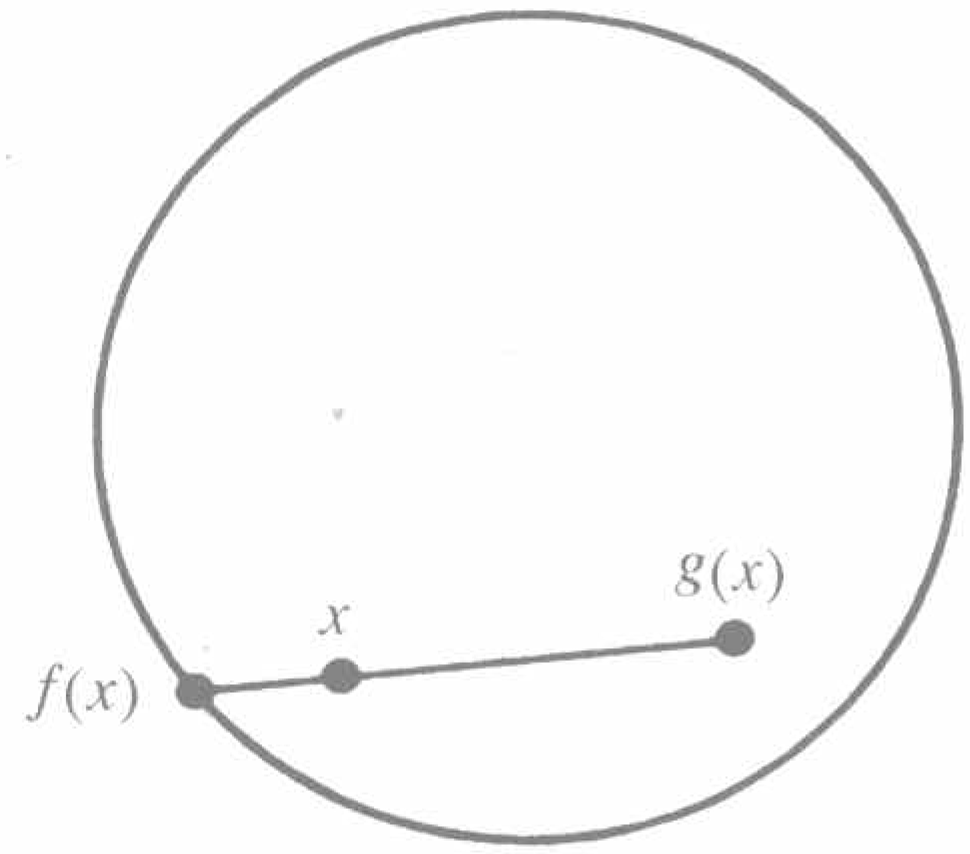



*Figure 4*

of $D^n$ into points outside of $D^n$. To correct this we set

$$P(x) = P_1(x)/(1 + \epsilon).$$

Then clearly $P$ maps $D^n$ into $D^n$ and $||P(x) - G(x)|| < 2\epsilon$ for $x \in D^n$.

Suppose that $G(x) \neq x$ for all $x \in D^n$. Then the continuous function $||G(x) - x||$ must take on a minimum $\mu > 0$ on $D^n$. Choosing $P : D^n \to D^n$ as above, with $||P(x) - G(x)|| < \mu$ for all $x$, we clearly have $P(x) \neq x$. Thus $P$ is a smooth map from $D^n$ to itself without a fixed point. This contradicts Lemma 6, and completes the proof.

The procedure employed here can frequently be applied in more general situations: to prove a proposition about continuous mappings, we first establish the result for smooth mappings and then try to use an approximation theorem to pass to the continuous case. (Compare §8, Problem 4.)

# §3. PROOF OF SARD'S THEOREM*

FIRST let us recall the statement:

**Theorem of Sard.** *Let $f : U \to R^p$ be a smooth map, with $U$ open in $R^n$, and let $C$ be the set of critical points; that is the set of all $x \varepsilon U$ with*

$$\text{rank } df_x < p.$$

*Then $f(C) \subset R^p$ has measure zero.*

REMARK. The cases where $n \leq p$ are comparatively easy. (Compare de Rham [29, p. 10].) We will, however, give a unified proof which makes these cases look just as bad as the others.

The proof will be by induction on $n$. Note that the statement makes sense for $n \geq 0$, $p \geq 1$. (By definition $R^0$ consists of a single point.) To start the induction, the theorem is certainly true for $n = 0$.

Let $C_1 \subset C$ denote the set of all $x \varepsilon U$ such that the first derivative $df_x$ is zero. More generally let $C_i$ denote the set of $x$ such that all partial derivatives of $f$ of order $\leq i$ vanish at $x$. Thus we have a descending sequence of closed sets

$$C \supset C_1 \supset C_2 \supset C_3 \supset \cdots .$$

The proof will be divided into three steps as follows:

STEP 1. The image $f(C - C_1)$ has measure zero.
STEP 2. The image $f(C_i - C_{i+1})$ has measure zero, for $i \geq 1$.
STEP 3. The image $f(C_k)$ has measure zero for $k$ sufficiently large.

(REMARK. If $f$ happens to be real analytic, then the intersection of

* Our proof is based on that given by Pontryagin [28]. The details are somewhat easier since we assume that $f$ is infinitely differentiable.

the $C_i$ is vacuous unless $f$ is constant on an entire component of $U$. Hence in this case it is sufficient to carry out Steps 1 and 2.)

PROOF OF STEP 1. This first step is perhaps the hardest. We may assume that $p \geq 2$, since $C = C_1$ when $p = 1$. We will need the well known theorem of Fubini* which asserts that *a measurable set*

$$A \subset R^p = R^1 \times R^{p-1}$$

*must have measure zero if it intersects each hyperplane* (constant) $\times R^{p-1}$ *in a set of* $(p-1)$*-dimensional measure zero.*

For each $\bar{x} \in C - C_1$ we will find an open neighborhood $V \subset R^n$ so that $f(V \cap C)$ has measure zero. Since $C - C_1$ is covered by countably many of these neighborhoods, this will prove that $f(C - C_1)$ has measure zero.

Since $\bar{x} \notin C_1$, there is some partial derivative, say $\partial f_1/\partial x_1$, which is not zero at $\bar{x}$. Consider the map $h : U \to R^n$ defined by

$$h(x) = (f_1(x), x_2, \cdots, x_n).$$

Since $dh_{\bar{x}}$ is nonsingular, $h$ maps some neighborhood $V$ of $\bar{x}$ diffeomorphically onto an open set $V'$. The composition $g = f \circ h^{-1}$ will then map $V'$ into $R^p$. Note that the set $C'$ of critical points of $g$ is precisely $h(V \cap C)$; hence the set $g(C')$ of critical values of $g$ is equal to $f(V \cap C)$.

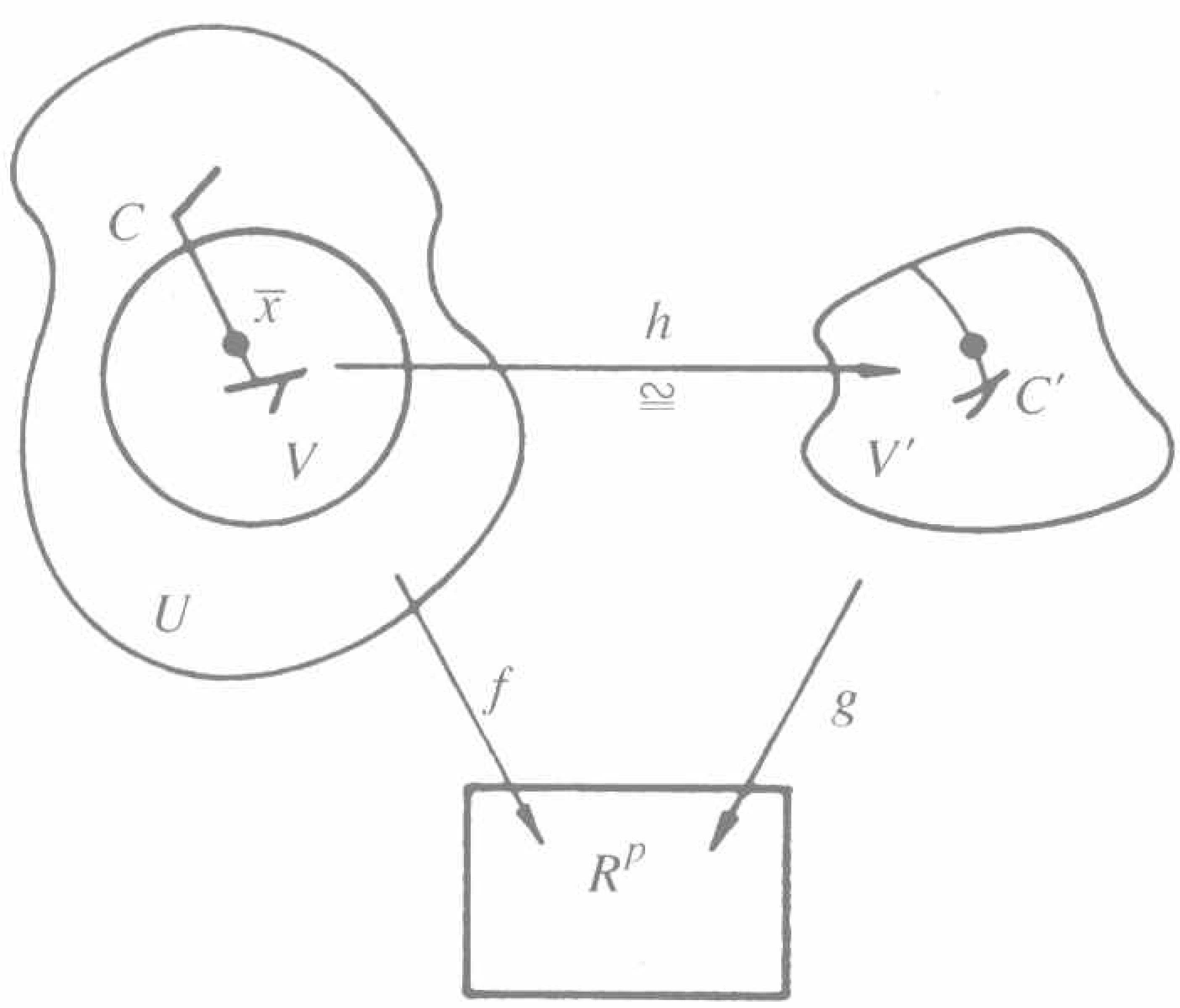



*Figure 5. Construction of the map g*

* For an easy proof (as well as an alternative proof of Sard's theorem) see Sternberg [35, pp. 51–52]. Sternberg assumes that $A$ is compact, but the general case follows easily from this special case.

For each $(t, x_2, \cdots, x_n) \in V'$ note that $g(t, x_2, \cdots, x_n)$ belongs to the hyperplane $t \times R^{p-1} \subset R^p$: thus $g$ carries hyperplanes into hyperplanes. Let

$$g^t : (t \times R^{n-1}) \cap V' \to t \times R^{p-1}$$

denote the restriction of $g$. Note that a point of $t \times R^{n-1}$ is critical for $g^t$ if and only if it is critical for $g$; for the matrix of first derivatives of $g$ has the form

$$(\partial g_i/\partial x_j) = \begin{bmatrix} 1 & 0 \\ * & (\partial g_i^t/\partial x_j) \end{bmatrix}.$$

According to the induction hypothesis, the set of critical values of $g^t$ has measure zero in $t \times R^{p-1}$. Therefore the set of critical values of $g$ intersects each hyperplane $t \times R^{p-1}$ in a set of measure zero. This set $g(C')$ is measurable, since it can be expressed as a countable union of compact subsets. Hence, by Fubini's theorem, the set

$$g(C') = f(V \cap C)$$

has measure zero, and Step 1 is complete.

Proof of Step 2. For each $\bar{x} \in C_k - C_{k+1}$ there is some $(k + 1) - st$ derivative $\partial^{k+1} f_r/\partial x_{s_1} \cdots \partial x_{s_{k+1}}$ which is not zero. Thus the function

$$w(x) = \partial^k f_r/\partial x_{s_2} \cdots \partial x_{s_{k+1}}$$

vanishes at $\bar{x}$ but $\partial w/\partial x_{s_1}$ does not. Suppose for definiteness that $s_1 = 1$. Then the map $h : U \to R^n$ defined by

$$h(x) = (w(x), x_2, \cdots, x_n)$$

carries some neighborhood $V$ of $\bar{x}$ diffeomorphically onto an open set $V'$. Note that $h$ carries $C_k \cap V$ into the hyperplane $0 \times R^{n-1}$. Again we consider

$$g = f \circ h^{-1} : V' \to R^p.$$

Let

$$\bar{g} : (0 \times R^{n-1}) \cap V' \to R^p$$

denote the restriction of $g$. By induction, the set of critical values of $\bar{g}$ has measure zero in $R^p$. But each point in $h(C_k \cap V)$ is certainly a critical point of $\bar{g}$ (since all derivatives of order $\leq k$ vanish). Therefore

$$\bar{g}h(C_k \cap V) = f(C_k \cap V) \text{ has measure zero.}$$

Since $C_k - C_{k+1}$ is covered by countably many such sets $V$, it follows that $f(C_k - C_{k+1})$ has measure zero.

PROOF OF STEP 3. Let $I^n \subset U$ be a cube with edge $\delta$. If $k$ is sufficiently large ($k > n/p - 1$ to be precise) we will prove that $f(C_k \cap I^n)$ has measure zero. Since $C_k$ can be covered by countably many such cubes, this will prove that $f(C_k)$ has measure zero.

From Taylor's theorem, the compactness of $I^n$, and the definition of $C_k$, we see that

$$f(x + h) = f(x) + R(x, h)$$

where

$$||R(x, h)|| \leq c\ ||h||^{k+1} \tag{1}$$

for $x \in C_k \cap I^n$, $x + h \in I^n$. Here $c$ is a constant which depends only on $f$ and $I^n$. Now subdivide $I^n$ into $r^n$ cubes of edge $\delta/r$. Let $I_1$ be a cube of the subdivision which contains a point $x$ of $C_k$. Then any point of $I_1$ can be written as $x + h$, with

$$||h|| \leq \sqrt{n}(\delta/r). \tag{2}$$

From 1) it follows that $f(I_1)$ lies in a cube of edge $a/r^{k+1}$ centered about $f(x)$, where $a = 2c\ (\sqrt{n}\ \delta)^{k+1}$ is constant. Hence $f(C_k \cap I^n)$ is contained in a union of at most $r^n$ cubes having total volume

$$V \leq r^n(a/r^{k+1})^p = a^p r^{n-(k+1)p}.$$

If $k + 1 > n/p$, then evidently $V$ tends to 0 as $r \to \infty$; so $f(C_k \cap I^n)$ must have measure zero. This completes the proof of Sard's theorem.

# §4. THE DEGREE MODULO 2 OF A MAPPING

CONSIDER a smooth map $f : S^n \to S^n$. If $y$ is a regular value, recall that $\#f^{-1}(y)$ denotes the number of solutions $x$ to the equation $f(x) = y$. We will prove that *the residue class modulo* 2 *of* $\#f^{-1}(y)$ *does not depend on the choice of the regular value* $y$. This residue class is called the mod 2 degree of $f$. More generally this same definition works for any smooth map

$$f : M \to N$$

where $M$ is compact without boundary, $N$ is connected, and both manifolds have the same dimension. (We may as well assume also that $N$ is compact without boundary, since otherwise the mod 2 degree would necessarily be zero.) For the proof we introduce two new concepts.

## SMOOTH HOMOTOPY AND SMOOTH ISOTOPY

Given $X \subset R^k$, let $X \times [0, 1]$ denote the subset* of $R^{k+1}$ consisting of all $(x, t)$ with $x \in X$ and $0 \leq t \leq 1$. Two mappings

$$f, g : X \to Y$$

are called *smoothly homotopic* (abbreviated $f \sim g$) if there exists a

* If $M$ is a smooth manifold without boundary, then $M \times [0, 1]$ is a smooth manifold bounded by two "copies" of $M$. Boundary points of $M$ will give rise to "corner" points of $M \times [0, 1]$.

smooth map $F : X \times [0, 1] \to Y$ with

$$F(x, 0) = f(x), \qquad F(x, 1) = g(x)$$

for all $x \in X$. This map $F$ is called a *smooth homotopy* between $f$ and $g$.

Note that the relation of smooth homotopy is an equivalence relation. To see that it is transitive we use the existence of a smooth function $\varphi : [0, 1] \to [0, 1]$ with

$$\varphi(t) = 0 \quad \text{for} \quad 0 \leq t \leq \tfrac{1}{3}$$

$$\varphi(t) = 1 \quad \text{for} \quad \tfrac{2}{3} \leq t \leq 1.$$

(For example, let $\varphi(t) = \lambda(t - \frac{1}{3})/(\lambda(t - \frac{1}{3}) + \lambda(\frac{2}{3} - t))$, where $\lambda(\tau) = 0$ for $\tau \leq 0$ and $\lambda(\tau) = \exp(-\tau^{-1})$ for $\tau > 0$.) Given a smooth homotopy $F$ between $f$ and $g$, the formula $G(x, t) = F(x, \varphi(t))$ defines a smooth homotopy $G$ with

$$G(x, t) = f(x) \quad \text{for} \quad 0 \leq t \leq \tfrac{1}{3}$$

$$G(x, t) = g(x) \quad \text{for} \quad \tfrac{2}{3} \leq t \leq 1.$$

Now if $f \sim g$ and $g \sim h$, then, with the aid of this construction, it is easy to prove that $f \sim h$.

If $f$ and $g$ happen to be diffeomorphisms from $X$ to $Y$, we can also define the concept of a "smooth isotopy" between $f$ and $g$. This also will be an equivalence relation.

DEFINITION. The diffeomorphism $f$ is *smoothly isotopic* to $g$ if there exists a smooth homotopy $F : X \times [0, 1] \to Y$ from $f$ to $g$ so that, for each $t \in [0, 1]$, the correspondence

$$x \mapsto F(x, t)$$

maps $X$ diffeomorphically onto $Y$.

It will turn out that the mod 2 degree of a map depends only on its smooth homotopy class:

**Homotopy Lemma.** *Let $f, g : M \to N$ be smoothly homotopic maps between manifolds of the same dimension, where $M$ is compact and without boundary. If $y \in N$ is a regular value for both $f$ and $g$, then*

$$\#f^{-1}(y) \equiv \#g^{-1}(y) \pmod 2.$$

PROOF. Let $F : M \times [0, 1] \to N$ be a smooth homotopy between $f$ and $g$. First suppose that $y$ is also a regular value for $F$. Then $F^{-1}(y)$

is a compact 1-manifold, with boundary equal to

$$F^{-1}(y) \cap (M \times 0 \cup M \times 1) = f^{-1}(y) \times 0 \cup g^{-1}(y) \times 1.$$

Thus the total number of boundary points of $F^{-1}(y)$ is equal to

$$\# f^{-1}(y) + \# g^{-1}(y).$$

But we recall from §2 that a compact 1-manifold always has an even number of boundary points. Thus $\# f^{-1}(y) + \# g^{-1}(y)$ is even, and therefore

$$\# f^{-1}(y) \equiv \# g^{-1}(y) \pmod 2.$$

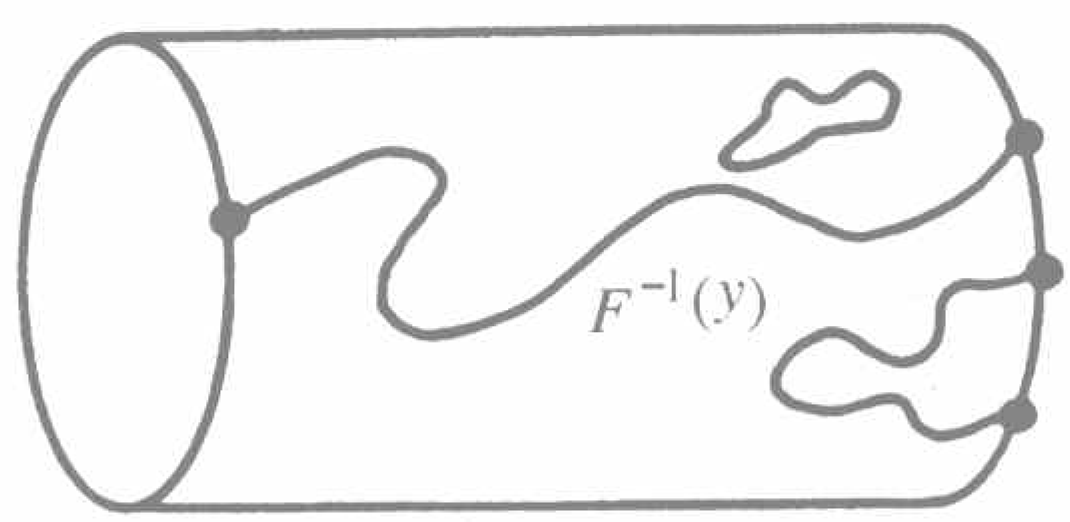



*Figure 6. The number of boundary points on the left is congruent to the number on the right modulo 2*

Now suppose that $y$ is not a regular value of $F$. Recall (from §1) that $\# f^{-1}(y')$ and $\# g^{-1}(y')$ are locally constant functions of $y'$ (as long as we stay away from critical values). Thus there is a neighborhood $V_1 \subset N$ of $y$, consisting of regular values of $f$, so that

$$\# f^{-1}(y') = \# f^{-1}(y)$$

for all $y' \in V_1$; and there is an analogous neighborhood $V_2 \subset N$ so that

$$\# g^{-1}(y') = \# g^{-1}(y)$$

for all $y' \in V_2$. Choose a regular value $z$ of $F$ within $V_1 \cap V_2$. Then

$$\# f^{-1}(y) = \# f^{-1}(z) \equiv \# g^{-1}(z) = \# g^{-1}(y),$$

which completes the proof.

We will also need the following:

**Homogeneity Lemma.** *Let $y$ and $z$ be arbitrary interior points of the smooth, connected manifold $N$. Then there exists a diffeomorphism $h: N \to N$ that is smoothly isotopic to the identity and carries $y$ into $z$.*

(For the special case $N = S^n$ the proof is easy: simply choose $h$ to be the rotation which carries $y$ into $z$ and leaves fixed all vectors orthogonal to the plane through $y$ and $z$.)

The proof in general proceeds as follows: We will first construct a smooth isotopy from $R^n$ to itself which

1) leaves all points outside of the unit ball fixed, and
2) slides the origin to any desired point of the open unit ball.

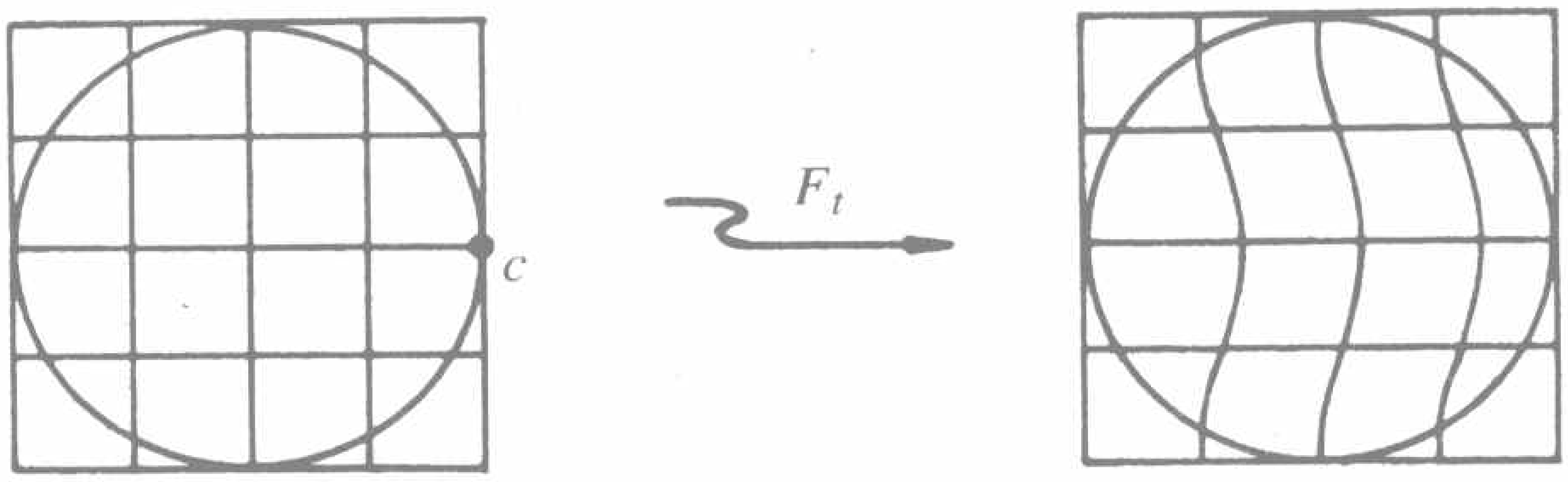



*Figure 7. Deforming the unit ball*

Let $\varphi : R^n \to R$ be a smooth function which satisfies

$$\varphi(x) > 0 \quad \text{for} \quad ||x|| < 1$$
$$\varphi(x) = 0 \quad \text{for} \quad ||x|| \geq 1.$$

(For example let $\varphi(x) = \lambda(1 - ||x||^2)$ where $\lambda(t) = 0$ for $t \leq 0$ and $\lambda(t) = \exp(-t^{-1})$ for $t > 0$.) Given any fixed unit vector $c \in S^{n-1}$, consider the differential equations

$$\frac{dx_i}{dt} = c_i\varphi(x_1, \cdots, x_n); \qquad i = 1, \cdots, n.$$

For any $\bar{x} \in R^n$ these equations have a unique solution $x = x(t)$, defined for all* real numbers, which satisfies the initial condition

$$x(0) = \bar{x}.$$

We will use the notation $x(t) = F_t(\bar{x})$ for this solution. Then clearly

1) $F_t(\bar{x})$ is defined for all $t$ and $\bar{x}$ and depends smoothly on $t$ and $\bar{x}$,
2) $F_0(\bar{x}) = \bar{x}$,
3) $F_{s+t}(\bar{x}) = F_s \circ F_t(\bar{x})$.

* Compare [22, §2.4].

Therefore each $F_t$ is a diffeomorphism from $R^n$ onto itself. Letting $t$ vary, we see that each $F_t$ is smoothly isotopic to the identity under an isotopy which leaves all points outside of the unit ball fixed. But clearly, with suitable choice of $c$ and $t$, the diffeomorphism $F_t$ will carry the origin to any desired point in the open unit ball.

Now consider a connected manifold $N$. Call two points of $N$ "isotopic" if there exists a smooth isotopy carrying one to the other. This is clearly an equivalence relation. If $y$ is an interior point, then it has a neighborhood diffeomorphic to $R^n$; hence the above argument shows that every point sufficiently close to $y$ is "isotopic" to $y$. In other words, each "isotopy class" of points in the interior of $N$ is an open set, and the interior of $N$ is partitioned into disjoint open isotopy classes. But the interior of $N$ is connected; hence there can be only one such isotopy class. This completes the proof.

We can now prove the main result of this section. Assume that $M$ is compact and boundaryless, that $N$ is connected, and that $f : M \to N$ is smooth.

**Theorem.** *If $y$ and $z$ are regular values of $f$ then*

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \quad \text{(modulo 2)}.$$

*This common residue class, which is called the* mod 2 degree *of $f$, depends only on the smooth homotopy class of $f$.*

Proof. Given regular values $y$ and $z$, let $h$ be a diffeomorphism from $N$ to $N$ which is isotopic to the identity and which carries $y$ to $z$. Then $z$ is a regular value of the composition $h \circ f$. Since $h \circ f$ is homotopic to $f$, the Homotopy Lemma asserts that

$$\#(h \circ f)^{-1}(z) \equiv \#f^{-1}(z) \quad \text{(mod 2)}.$$

But

$$(h \circ f)^{-1}(z) = f^{-1}h^{-1}(z) = f^{-1}(y),$$

so that

$$\#(h \circ f)^{-1}(z) = \#f^{-1}(y).$$

Therefore

$$\#f^{-1}(y) \equiv \#f^{-1}(z) \quad \text{(mod 2)},$$

as required.

Call this common residue class $\deg_2(f)$. Now suppose that $f$ is smoothly homotopic to $g$. By Sard's theorem, there exists an element $y \in N$

which is a regular value for both $f$ and $g$. The congruence

$$\deg_2 f \equiv \# f^{-1}(y) \equiv \# g^{-1}(y) \equiv \deg_2 g \qquad (\text{mod } 2)$$

now shows that $\deg_2 f$ is a smooth homotopy invariant, and completes the proof.

EXAMPLES. A constant map $c : M \to M$ has even mod 2 degree. The identity map $I$ of $M$ has odd degree. *Hence the identity map of a compact boundaryless manifold is not homotopic to a constant.*

In the case $M = S^n$, this result implies the assertion that no smooth map $f : D^{n+1} \to S^n$ leaves the sphere pointwise fixed. (I.e., the sphere is not a smooth "retract" of the disk. Compare §2, Lemma 5.) For such a map $f$ would give rise to a smooth homotopy

$$F : S^n \times [0, 1] \to S^n, \qquad F(x, t) = f(tx),$$

between a constant map and the identity.

# §5. ORIENTED MANIFOLDS

IN ORDER to define the degree as an integer (rather than an integer modulo 2) we must introduce orientations.

DEFINITIONS. An orientation for a finite dimensional real vector space is an equivalence class of ordered bases as follows: the ordered basis $(b_1, \cdots, b_n)$ determines the *same orientation* as the basis $(b'_1, \cdots, b'_n)$ if $b'_i = \sum a_{ij}b_j$ with $\det(a_{ij}) > 0$. It determines the *opposite orientation* if $\det(a_{ij}) < 0$. Thus each positive dimensional vector space has precisely two orientations. The vector space $R^n$ has a *standard* orientation corresponding to the basis $(1, 0, \cdots, 0), (0, 1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)$.

In the case of the zero dimensional vector space it is convenient to define an "orientation" as the symbol $+1$ or $-1$.

An *oriented* smooth manifold consists of a manifold $M$ together with a choice of orientation for each tangent space $TM_x$. If $m \geq 1$, these are required to fit together as follows: For each point of $M$ there should exist a neighborhood $U \subset M$ and a diffeomorphism $h$ mapping $U$ onto an open subset of $R^m$ or $H^m$ which is *orientation preserving*, in the sense that for each $x \in U$ the isomorphism $dh_x$ carries the specified orientation for $TM_x$ into the standard orientation for $R^m$.

If $M$ is connected and orientable, then it has precisely two orientations.

If $M$ has a boundary, we can distinguish three kinds of vectors in the tangent space $TM_x$ at a boundary point:

1) there are the vectors tangent to the boundary, forming an $(m - 1)$-dimensional subspace $T(\partial M)_x \subset TM_x$;

2) there are the "outward" vectors, forming an open half space bounded by $T(\partial M)_x$;

3) there are the "inward" vectors forming a complementary half space.

Each orientation for $M$ determines an orientation for $\partial M$ as follows: For $x \in \partial M$ choose a positively oriented basis $(v_1, v_2, \cdots, v_m)$ for $TM_x$ in such a way that $v_2, \cdots, v_m$ are tangent to the boundary (assuming that $m \geq 2$) and that $v_1$ is an "outward" vector. Then $(v_2, \cdots, v_m)$ determines the required orientation for $\partial M$ at $x$.

If the dimension of $M$ is 1, then each boundary point $x$ is assigned the orientation $-1$ or $+1$ according as a positively oriented vector at $x$ points inward or outward. (See Figure 8.)

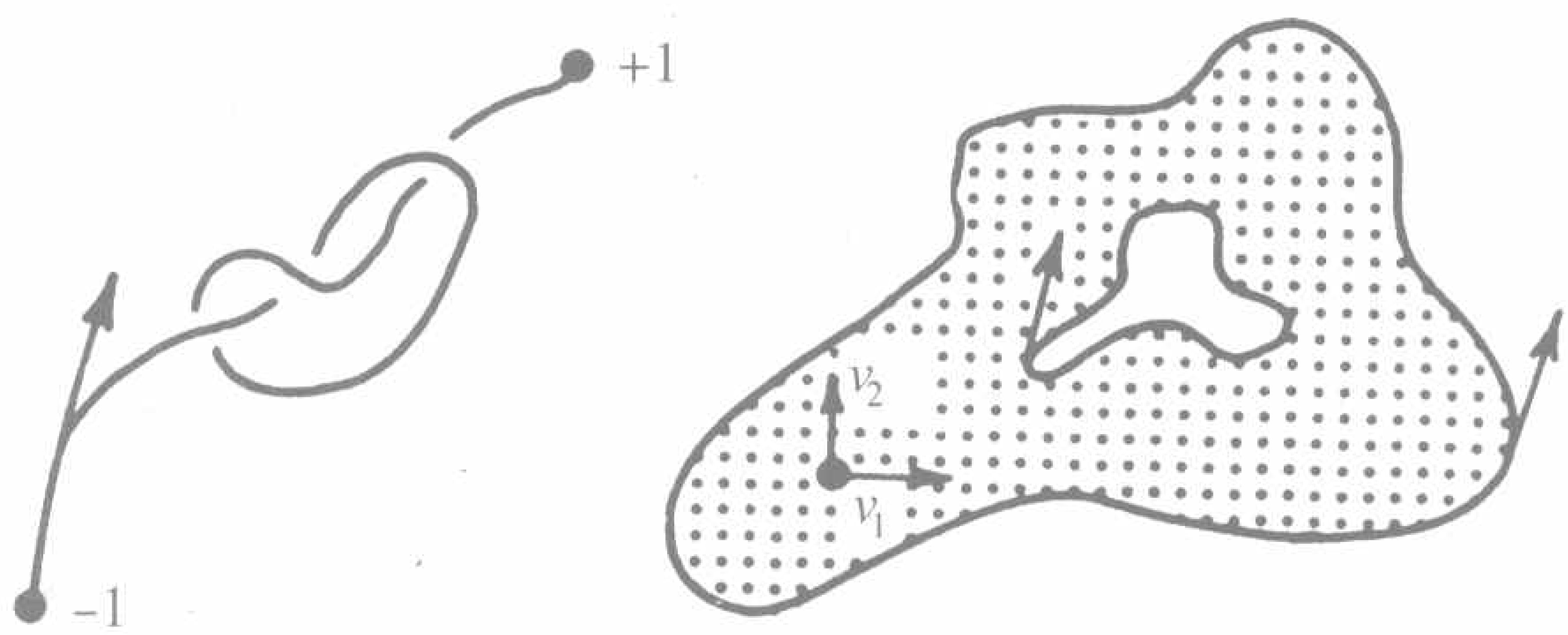



*Figure 8. How to orient a boundary*

As an example the unit sphere $S^{m-1} \subset R^m$ can be oriented as the boundary of the disk $D^m$.

## THE BROUWER DEGREE

Now let $M$ and $N$ be oriented $n$-dimensional manifolds without boundary and let

$$f : M \to N$$

be a smooth map. If $M$ is compact and $N$ is connected, then the degree of $f$ is defined as follows:

Let $x \in M$ be a regular point of $f$, so that $df_x : TM_x \to TN_{f(x)}$ is a linear isomorphism between oriented vector spaces. Define the *sign* of $df_x$ to be $+1$ or $-1$ according as $df_x$ preserves or reverses orientation. For any regular value $y \in N$ define

$$\deg(f; y) = \sum_{x \in f^{-1}(y)} \operatorname{sign} df_x .$$

As in §1, this integer $\deg(f; y)$ is a locally constant function of $y$. It is defined on a dense open subset of $N$.

**Theorem A.** *The integer deg(f; y) does not depend on the choice of regular value y.*

It will be called the *degree* of $f$ (denoted $\deg f$).

**Theorem B.** *If f is smoothly homotopic to g, then* $\deg f = \deg g$.

The proof will be essentially the same as that in §4. It is only necessary to keep careful control of orientations.

First consider the following situation: Suppose that $M$ is the boundary of a compact oriented manifold $X$ and that $M$ is oriented as the boundary of $X$.

**Lemma 1.** *If* $f : M \to N$ *extends to a smooth map* $F : X \to N$, *then* $\deg(f; y) = 0$ *for every regular value y.*

Proof. First suppose that $y$ is a regular value for $F$, as well as for $f = F \mid M$. The compact 1-manifold $F^{-1}(y)$ is a finite union of arcs and circles, with only the boundary points of the arcs lying on $M = \partial X$. Let $A \subset F^{-1}(y)$ be one of these arcs, with $\partial A = \{a\} \cup \{b\}$. We will show that

$$\text{sign } df_a + \text{sign } df_b = 0,$$

and hence (summing over all such arcs) that $\deg(f\,; y) = 0$.

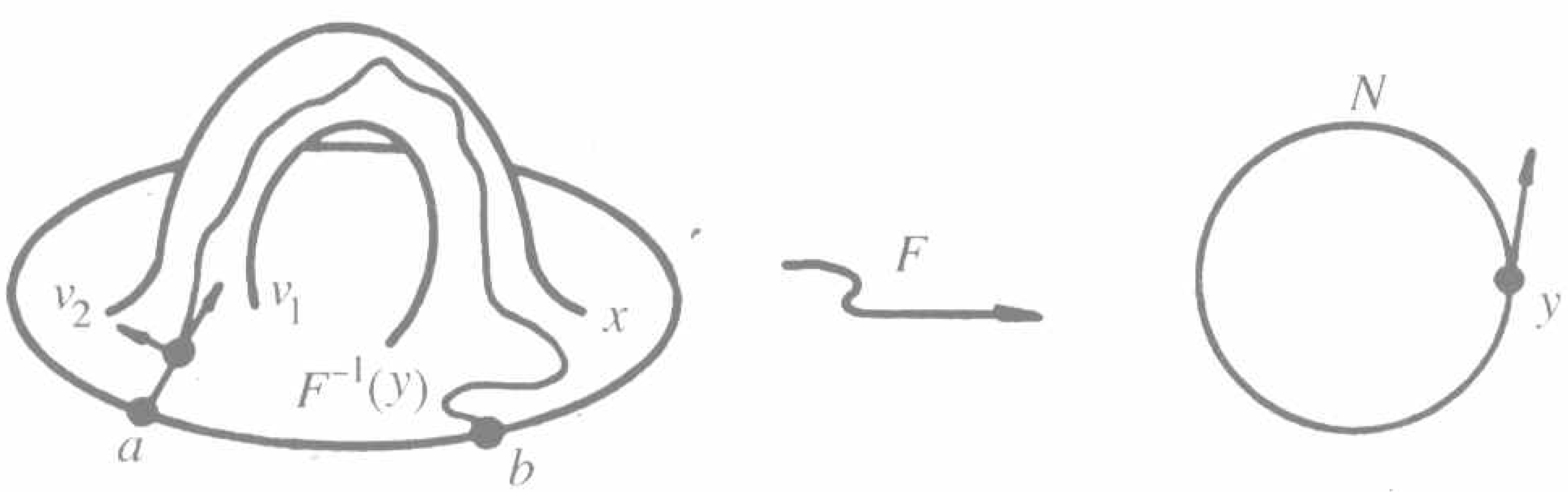



*Figure 9. How to orient* $F^{-1}(y)$

The orientations for $X$ and $N$ determine an orientation for $A$ as follows: Given $x \in A$, let $(v_1, \cdots, v_{n+1})$ be a positively oriented basis for $TX_x$ with $v_1$ tangent to $A$. *Then* $v_1$ *determines the required orientation for* $TA_x$ *if and only if* $dF_x$ *carries* $(v_2, \cdots, v_{n+1})$ *into a positively oriented basis for* $TN_y$.

Let $v_1(x)$ denote the positively oriented unit vector tangent to $A$ at $x$. Clearly $v_1$ is a smooth function, and $v_1(x)$ points outward at one boundary point (say $b$) and inward at the other boundary point $a$.

It follows immediately that

$$\text{sign } df_a = -1, \qquad \text{sign } df_b = +1;$$

with sum zero. Adding up over all such arcs $A$, we have proved that $\deg(f; y) = 0$.

More generally, suppose that $y_0$ is a regular value for $f$, but not for $F$. The function $\deg(f; y)$ is constant within some neighborhood $U$ of $y_0$. Hence, as in §4, we can choose a regular value $y$ for $F$ within $U$ and observe that

$$\deg(f; y_0) = \deg(f; y) = 0.$$

This proves Lemma 1.

Now consider a smooth homotopy $F : [0, 1] \times M \to N$ between two mappings $f(x) = F(0, x)$, $g(x) = F(1, x)$.

**Lemma 2.** *The degree* $\deg(g; y)$ *is equal to* $\deg(f; y)$ *for any common regular value* $y$.

Proof. The manifold $[0, 1] \times M^n$ can be oriented as a product, and will then have boundary consisting of $1 \times M^n$ (with the correct orientation) and $0 \times M^n$ (with the wrong orientation). Thus the degree of $F \mid \partial([0, 1] \times M^n)$ at a regular value $y$ is equal to the difference

$$\deg(g; y) - \deg(f; y).$$

According to Lemma 1 this difference must be zero.

The remainder of the proof of Theorems A and B is completely analogous to the argument in §4. If $y$ and $z$ are both regular values for $f : M \to N$, choose a diffeomorphism $h : N \to N$ that carries $y$ to $z$ and is isotopic to the identity. Then $h$ will preserve orientation, and

$$\deg(f; y) = \deg(h \circ f; h(y))$$

by inspection. But $f$ is homotopic to $h \circ f$; hence

$$\deg(h \circ f; z) = \deg(f; z)$$

by Lemma 2. Therefore $\deg(f; y) = \deg(f; z)$, which completes the proof.

Examples. The complex function $z \to z^k$, $z \neq 0$, maps the unit circle onto itself with degree $k$. (Here $k$ may be positive, negative, or zero.) The degenerate mapping

$$f : M \to \text{constant} \in N$$

has degree zero. A diffeomorphism $f : M \to N$ has degree $+1$ or $-1$ according as $f$ preserves or reverses orientation. *Thus an orientation reversing diffeomorphism of a compact boundaryless manifold is not smoothly homotopic to the identity.*

One example of an orientation reversing diffeomorphism is provided by the reflection $r_i : S^n \to S^n$, where

$$r_i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, -x_i, \cdots, x_{n+1}).$$

The antipodal map of $S^n$ has degree $(-1)^{n+1}$, as we can see by noting that the antipodal map is the composition of $n + 1$ reflections:

$$-x = r_1 \circ r_2 \circ \cdots \circ r_{n+1}(x).$$

*Thus if $n$ is even, the antipodal map of $S^n$ is not smoothly homotopic to the identity*, a fact not detected by the degree modulo 2.

As an application, following Brouwer, we show that $S^n$ *admits a smooth field of nonzero tangent vectors if and only if $n$ is odd.* (Compare Figures 10 and 11.)

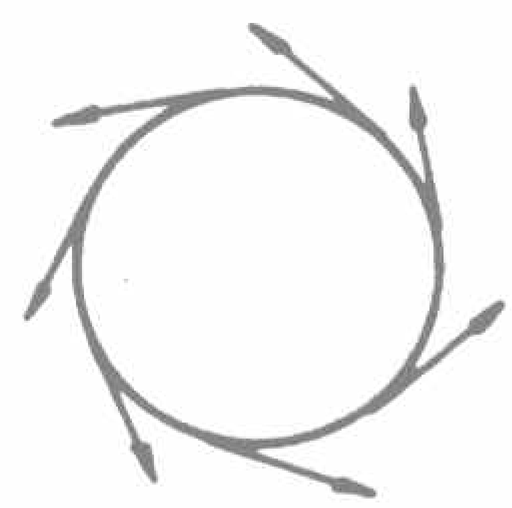
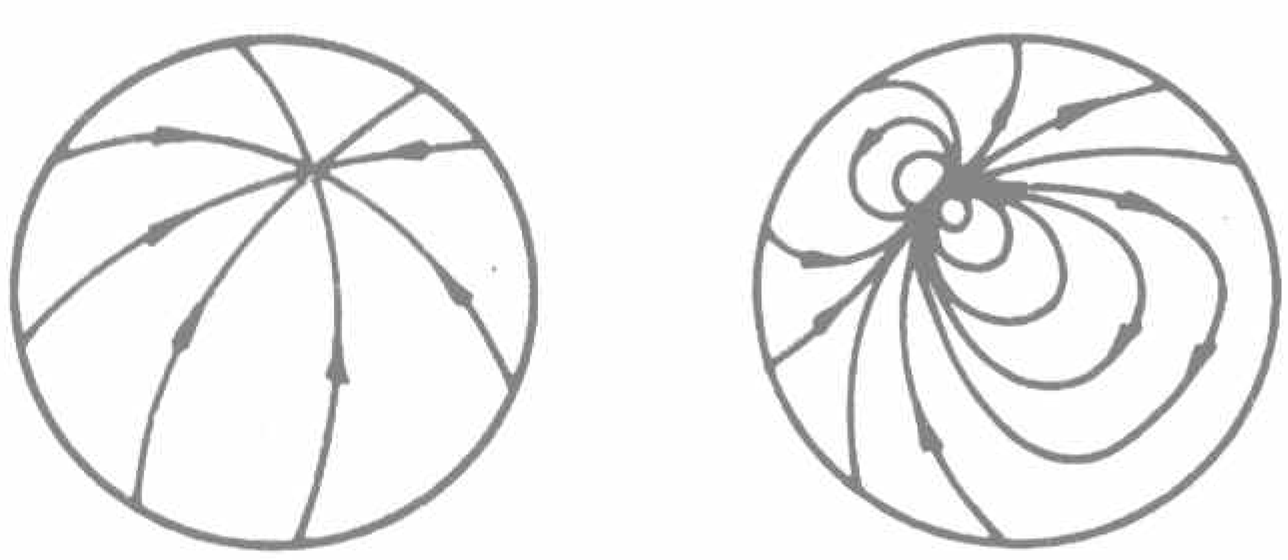


*Figure 10 (above). A nonzero vector field on the 1-sphere*

*Figure 11 (below). Attempts for $n = 2$*



DEFINITION. A smooth *tangent vector field* on $M \subset R^k$ is a smooth map $v : M \to R^k$ such that $v(x) \in TM_x$ for each $x \in M$. In the case of the sphere $S^n \subset R^{n+1}$ this is clearly equivalent to the condition

1) $$v(x) \cdot x = 0 \quad \text{for all} \quad x \in S^n,$$

using the euclidean inner product.

If $v(x)$ is nonzero for all $x$, then we may as well suppose that

2) $$v(x) \cdot v(x) = 1 \quad \text{for all} \quad x \in S^n.$$

For in any case $\bar{v}(x) = v(x)/||v(x)||$ would be a vector field which does satisfy this condition. Thus we can think of $v$ as a smooth function from $S^n$ to itself.

Now define a smooth homotopy

$$F : S^n \times [0, \pi] \to S^n$$

by the formula $F(x, \theta) = x \cos \theta + v(x) \sin \theta$. Computation shows that

$$F(x, \theta) \cdot F(x, \theta) = 1$$

and that

$$F(x, 0) = x, \qquad F(x, \pi) = -x.$$

Thus the antipodal map of $S^n$ is homotopic to the identity. But for $n$ even we have seen that this is impossible.

On the other hand, if $n = 2k - 1$, the explicit formula

$$v(x_1, \cdots, x_{2k}) = (x_2, -x_1, x_4, -x_3, \cdots, x_{2k}, -x_{2k-1})$$

defines a nonzero tangent vector field on $S^n$. This completes the proof.

It follows, incidentally, that the antipodal map of $S^n$ *is* homotopic to the identity for $n$ odd. A famous theorem due to Heinz Hopf asserts that two mappings from a connected $n$-manifold to the $n$-sphere are smoothly homotopic *if and only if* they have the same degree. In §7 we will prove a more general result which implies Hopf's theorem.

# §6. VECTOR FIELDS AND THE EULER NUMBER

As a further application of the concept of degree, we study vector fields on other manifolds.

Consider first an open set $U \subset R^m$ and a smooth vector field

$$v : U \to R^m$$

with an isolated zero at the point $z \in U$. The function

$$\bar{v}(x) = v(x)/||v(x)||$$

maps a small sphere centered at $z$ into the unit sphere.* The degree of this mapping is called the *index* $\iota$ of $v$ at the zero $z$.

Some examples, with indices $-1$, 0, 1, 2, are illustrated in Figure 12. (Intimately associated with $v$ are the curves "tangent" to $v$ which are obtained by solving the differential equations $dx_i/dt = v_i(x_1, \cdots, x_n)$. It is these curves which are actually sketched in Figure 12.)

A zero with arbitrary index can be obtained as follows: In the plane of complex numbers the polynomial $z^k$ defines a smooth vector field with a zero of index $k$ at the origin, and the function $\bar{z}^k$ defines a vector field with a zero of index $-k$.

We must prove that this concept of index is invariant under diffeomorphism of $U$. To explain what this means, let us consider the more general situation of a map $f : M \to N$, with a vector field on each manifold.

DEFINITION. The vector fields $v$ on $M$ and $v'$ on $N$ *correspond* under $f$ if the derivative $df_x$ carries $v(x)$ into $v'(f(x))$ for each $x \in M$.

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* Each sphere is to be oriented as the boundary of the corresponding disk.

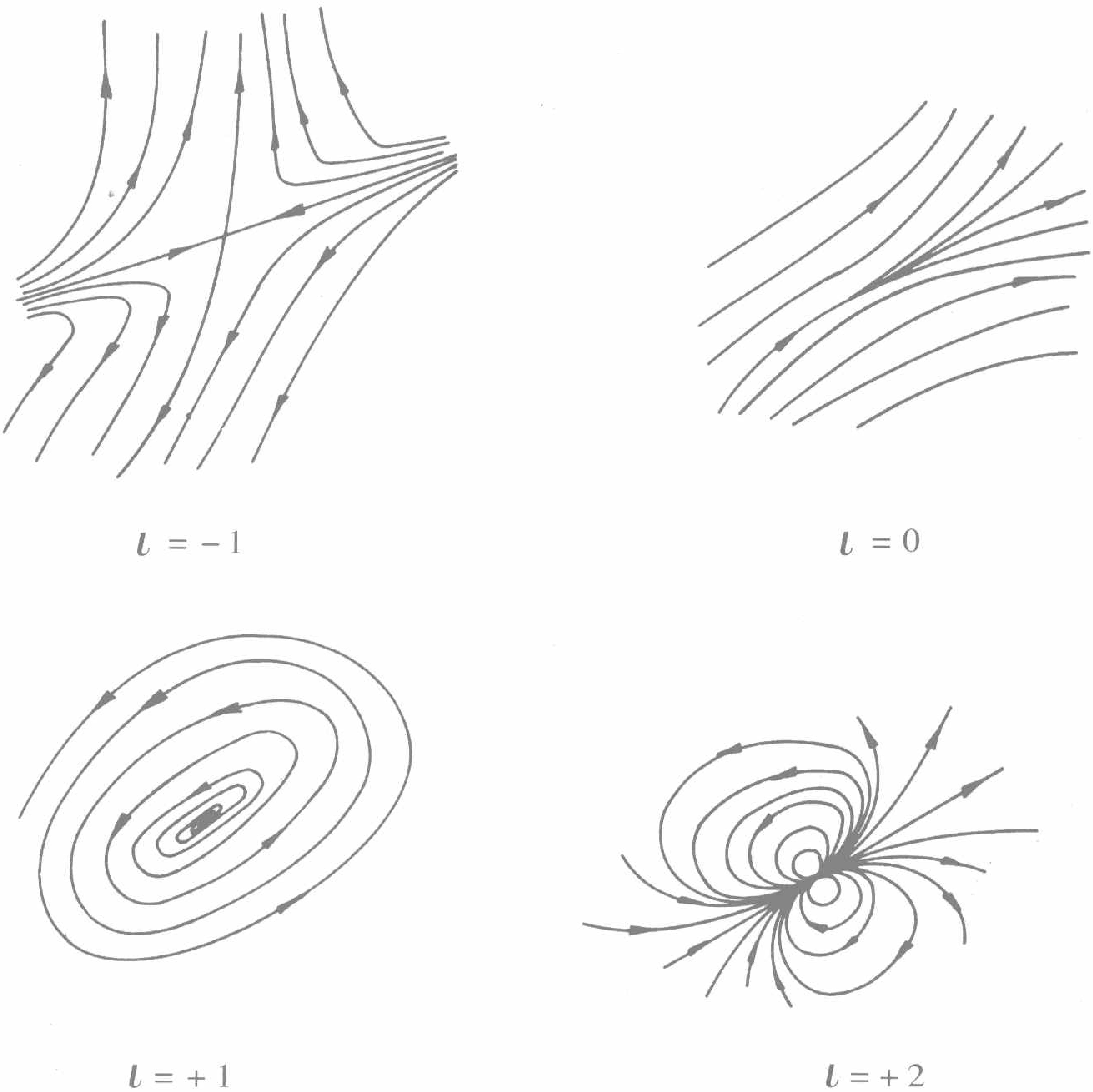


*Figure 12. Examples of plane vector fields*

If $f$ is a diffeomorphism, then clearly $v'$ is uniquely determined by $v$. The notation

$$v' = df \circ v \circ f^{-1}$$

will be used.

**Lemma 1.** *Suppose that the vector field $v$ on $U$ corresponds to*

$$v' = df \circ v \circ f^{-1}$$

*on $U'$ under a diffeomorphism $f : U \to U'$. Then the index of $v$ at an isolated zero $z$ is equal to the index of $v'$ at $f(z)$.*

Assuming Lemma 1, we can define the concept of index for a vector field $w$ on an arbitrary manifold $M$ as follows: If $g : U \to M$ is a parametrization of a neighborhood of $z$ in $M$, then the *index* $\iota$ of $w$ at $z$ is defined to be equal to the index of the corresponding vector field $dg^{-1} \circ w \circ g$ on $U$ at the zero $g^{-1}(z)$. It clearly will follow from Lemma 1 that $\iota$ is well defined.

The proof of Lemma 1 will be based on the proof of a quite different result:

**Lemma 2.** *Any orientation preserving diffeomorphism $f$ of $R^m$ is smoothly isotopic to the identity.*

(In contrast, for many values of $m$ there exists an orientation preserving diffeomorphism of the sphere $S^m$ which is not smoothly isotopic to the identity. See [20, p. 404].)

PROOF. We may assume that $f(0) = 0$. Since the derivative at 0 can be defined by

$$df_0(x) = \lim_{t \to 0} f(tx)/t,$$

it is natural to define an isotopy

$$F : R^m \times [0, 1] \to R^m$$

by the formula

$$F(x, t) = f(tx)/t \quad \text{for} \quad 0 < t \leq 1,$$

$$F(x, 0) = df_0(x).$$

To prove that $F$ is smooth, even as $t \to 0$, we write $f$ in the form*

$$f(x) = x_1 g_1(x) + \cdots + x_m g_m(x),$$

where $g_1, \cdots, g_m$ are suitable smooth functions, and note that

$$F(x, t) = x_1 g_1(tx) + \cdots + x_m g_m(tx)$$

for all values of $t$.

Thus $f$ is isotopic to the linear mapping $df_0$, which is clearly isotopic to the identity. This proves Lemma 2.

PROOF OF LEMMA 1. We may assume that $z = f(z) = 0$ and that $U$ is convex. If $f$ preserves orientation, then, proceeding exactly as above,

* See for example [22, p. 5].

we construct a one-parameter family of embeddings

$$f_t : U \to R^m$$

with $f_0$ = identity, $f_1 = f$, and $f_t(0) = 0$ for all $t$. Let $v_t$ denote the vector field $df_t \circ v \circ f_t^{-1}$ on $f_t(U)$, which corresponds to $v$ on $U$. These vector fields are all defined and nonzero on a sufficiently small sphere centered at 0. Hence the index of $v = v_0$ at 0 must be equal to the index of $v' = v_1$ at 0. This proves Lemma 1 for orientation preserving diffeomorphisms.

To consider diffeomorphisms which reverse orientation it is sufficient to consider the special case of a reflection $\rho$. Then

$$v' = \rho \circ v \circ \rho^{-1},$$

so the associated function $\bar{v}'(x) = v'(x)/||v'(x)||$ on the $\epsilon$-sphere satisfies

$$\bar{v}' = \rho \circ \bar{v} \circ \rho^{-1}.$$

Evidently the degree of $\bar{v}'$ equals the degree of $\bar{v}$, which completes the proof of Lemma 1.

We will study the following classical result: Let $M$ be a compact manifold and $w$ a smooth vector field on $M$ with isolated zeros. *If $M$ has a boundary, then $w$ is required to point outward at all boundary points.*

**Poincaré-Hopf Theorem.** *The sum $\sum \iota$ of the indices at the zeros of such a vector field is equal to the Euler number**

$$\chi(M) = \sum_{i=0}^{m} (-1)^i \ \text{rank} \ H_i(M).$$

*In particular this index sum is a topological invariant of $M$: it does not depend on the particular choice of vector field.*

(A 2-dimensional version of this theorem was proved by Poincare in 1885. The full theorem was proved by Hopf [14] in 1926 after earlier partial results by Brouwer and Hadamard.)

We will prove part of this theorem, and sketch a proof of the rest. First consider the special case of a compact domain in $R^m$.

Let $X \subset R^m$ be a compact $m$-manifold with boundary. The *Gauss mapping*

$$g : \partial X \to S^{m-1}$$

assigns to each $x \in \partial X$ the outward unit normal vector at $x$.

---

* Here $H_i(M)$ denotes the $i$-th homology group of $M$. This will be our first and last reference to homology theory.

**Lemma 3** (Hopf). *If $v : X \to R^m$ is a smooth vector field with isolated zeros, and if $v$ points out of $X$ along the boundary, then the index sum $\sum \iota$ is equal to the degree of the Gauss mapping from $\partial X$ to $S^{m-1}$. In particular, $\sum \iota$ does not depend on the choice of $v$.*

For example, if a vector field on the disk $D^m$ points outward along the boundary, then $\sum \iota = +1$. (Compare Figure 13.)

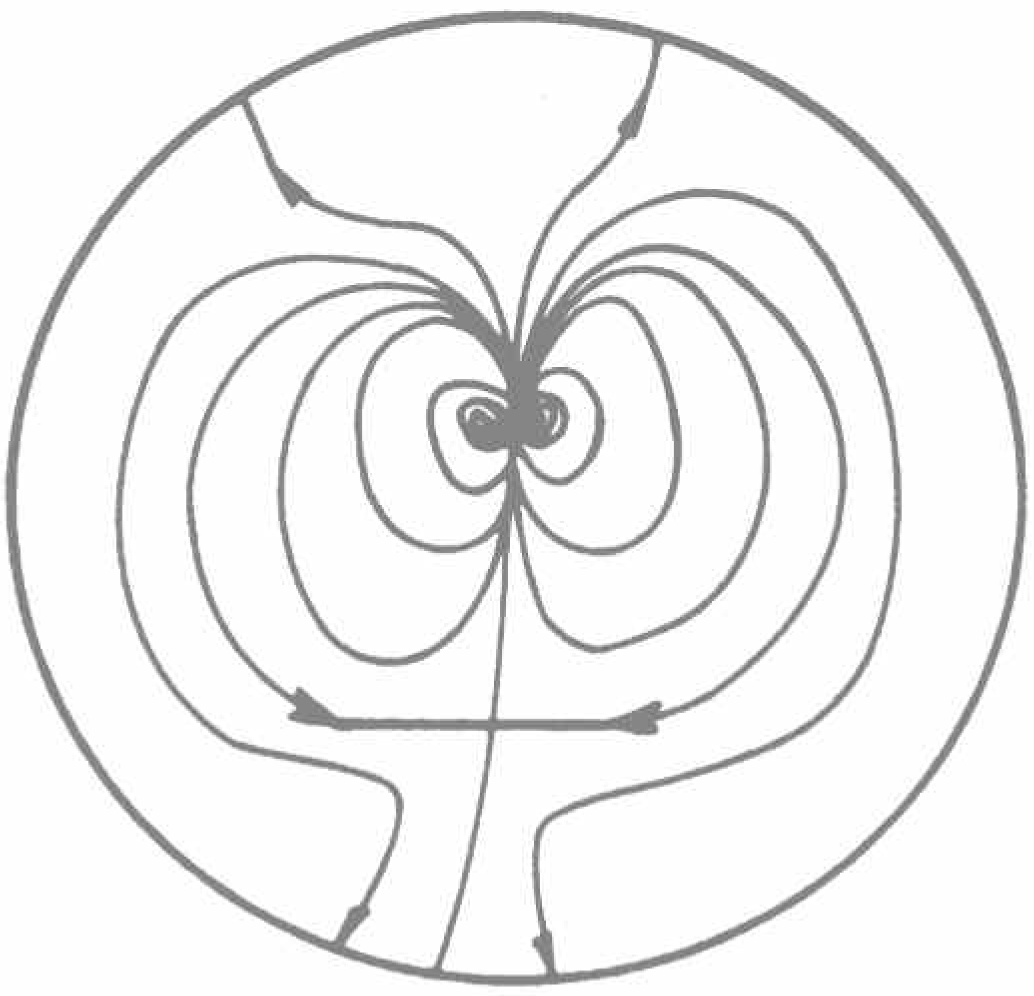

*Figure 13. An example with index sum +1*

PROOF. Removing an $\epsilon$-ball around each zero, we obtain a new manifold with boundary. The function $\bar{v}(x) = v(x)/||v(x)||$ maps this manifold into $S^{m-1}$. Hence the sum of the degrees of $\bar{v}$ restricted to the various boundary components is zero. But $\bar{v} \mid \partial X$ is homotopic to $g$, and the degrees on the other boundary components add up to $-\sum \iota$. (The minus sign occurs since each small sphere gets the wrong orientation.) Therefore

$$\deg(g) - \sum \iota = 0$$

as required.

REMARK. The degree of $g$ is also known as the "curvatura integra" of $\partial X$, since it can be expressed as a constant times the integral over $\partial X$ of the Gaussian curvature. This integer is of course equal to the Euler number of $X$. For $m$ odd it is equal to half the Euler number of $\partial X$.

Before extending this result to other manifolds, some more preliminaries are needed.

It is natural to try to compute the index of a vector field $v$ at a zero $z$

in terms of the derivatives of $v$ at $z$. Consider first a vector field $v$ on an open set $U \subset R^m$ and think of $v$ as a mapping $U \to R^m$, so that $dv_z : R^m \to R^m$ is defined.

DEFINITION. The vector field $v$ is *nondegenerate* at $z$ if the linear transformation $dv_z$ is nonsingular.

It follows that $z$ is an isolated zero.

**Lemma 4.** *The index of $v$ at a nondegenerate zero $z$ is either $+1$ or $-1$ according as the determinant of $dv_z$ is positive or negative.*

PROOF. Think of $v$ as a diffeomorphism from some convex neighborhood $U_0$ of $z$ into $R^m$. We may assume that $z = 0$. If $v$ preserves orientation, we have seen that $v|U_0$ can be deformed smoothly into the identity without introducing any new zeros. (See Lemmas 1, 2.) Hence the index is certainly equal to $+1$.

If $v$ reverses orientation, then similarly $v$ can be deformed into a reflection; hence $\iota = -1$.

More generally consider a zero $z$ of a vector field $w$ on a manifold $M \subset R^k$. Think of $w$ as a map from $M$ to $R^k$ so that the derivative $dw_z : TM_z \to R^k$ is defined.

**Lemma 5.** *The derivative $dw_z$ actually carries $TM_z$ into the subspace $TM_z \subset R^k$, and hence can be considered as a linear transformation from $TM_z$ to itself. If this linear transformation has determinant $D \neq 0$ then $z$ is an isolated zero of $w$ with index equal to $+1$ or $-1$ according as $D$ is positive or negative.*

PROOF. Let $h : U \to M$ be a parametrization of some neighborhood of $z$. Let $e^i$ denote the $i$-th basis vector of $R^m$ and let

$$t^i = dh_u(e^i) = \partial h/\partial u_i$$

so that the vectors $t^1, \cdots, t^m$ form a basis for the tangent space $TM_{h(u)}$. We must compute the image of $t^i = t^i(u)$ under the linear transformation $dw_{h(u)}$. First note that

$$\text{1)} \qquad dw_{h(u)}(t^i) = d(w \circ h)_u(e^i) = \partial w(h(u))/\partial u_i.$$

Let $v = \sum v_j e^j$ be the vector field on $U$ which corresponds to the vector field $w$ on $M$. By definition $v = dh^{-1} \circ w \circ h$, so that

$$w(h(u)) = dh_u(v) = \sum v_j t^j.$$

Therefore

$$\text{2)} \qquad \partial w(h(u))/\partial u_i = \textstyle\sum_j (\partial v_j/\partial u_i) t^j + \sum_j v_j (\partial t^j/\partial u_i).$$

Combining 1) and 2), and then evaluating at the zero $h^{-1}(z)$ of $v$, we obtain the formula

3) $$dw_z(t^i) = \sum_j (\partial v_j/\partial u_i)t^j.$$

Thus $dw_z$ maps $TM_z$ into itself, and the determinant $D$ of this linear transformation $TM_z \to TM_z$ is equal to the determinant of the matrix $(\partial v_j/\partial u_i)$. Together with Lemma 4 this completes the proof.

Now consider a compact, boundaryless manifold $M \subset R^k$. Let $N_\epsilon$ denote the closed $\epsilon$-neighborhood of $M$ (i.e., the set of all $x \in R^k$ with $\|x - y\| \leq \epsilon$ for some $y \in M$). For $\epsilon$ sufficiently small one can show that $N_\epsilon$ is a smooth manifold with boundary. (See §8, Problem 11.)

**Theorem 1.** *For any vector field $v$ on $M$ with only nondegenerate zeros, the index sum $\sum \iota$ is equal to the degree of the Gauss mapping**

$$g : \partial N_\epsilon \to S^{k-1}.$$

*In particular this sum does not depend on the choice of vector field.*

Proof. For $x \in N_\epsilon$ let $r(x) \in M$ denote the closest point of $M$. (Compare §8, Problem 12.) Note that the vector $x - r(x)$ is perpendicular to the tangent space of $M$ at $r(x)$, for otherwise $r(x)$ would not be the closest point of $M$. If $\epsilon$ is sufficiently small, then the function $r(x)$ is smooth and well defined.

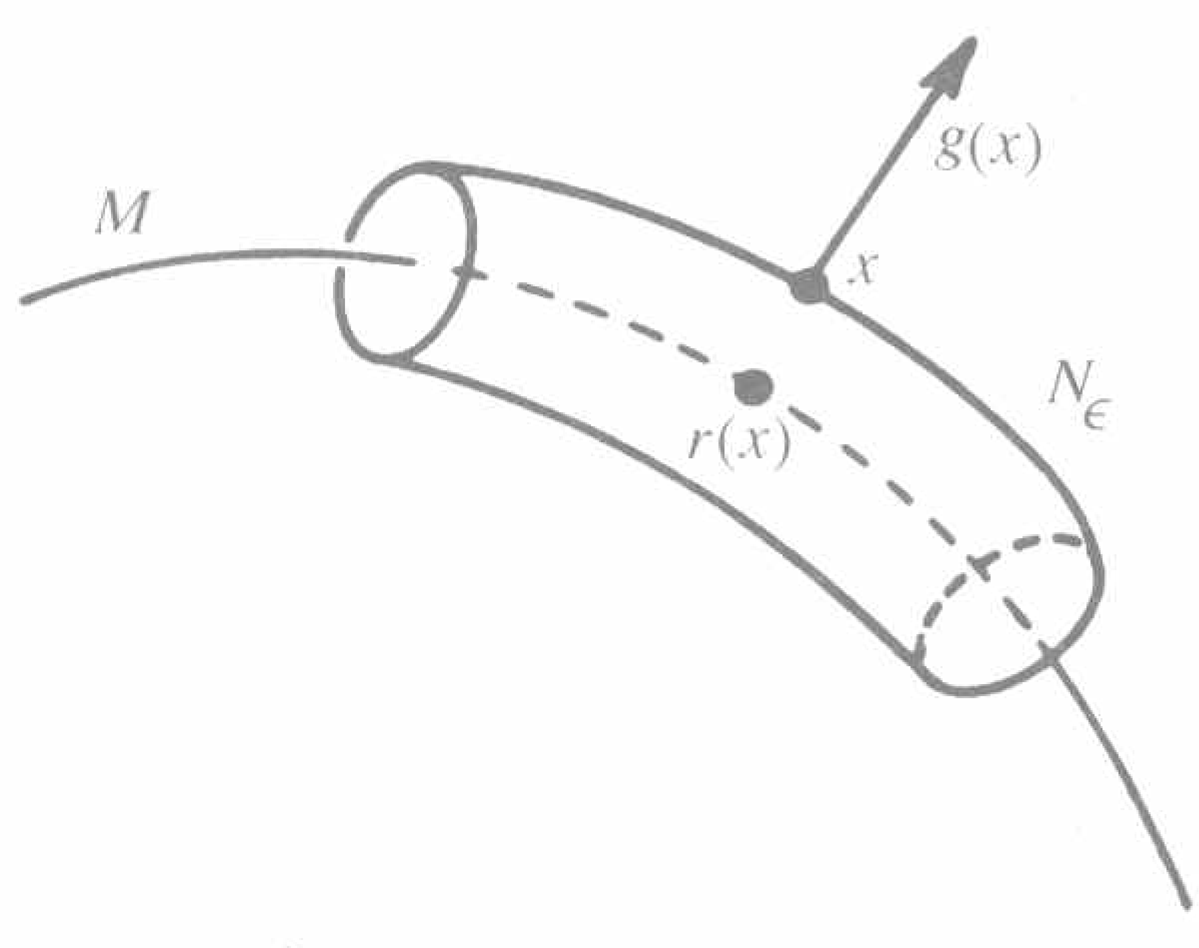


*Figure 14. The $\epsilon$-neighborhood of M*

* A different interpretation of this degree has been given by Allendoerfer and Fenchel: the degree of $g$ can be expressed as the integral over $M$ of a suitable curvature scalar, thus yielding an $m$-dimensional version of the classical Gauss-Bonnet theorem. (References [1], [9]. See also Chern [6].)

We will also consider the squared distance function

$$\varphi(x) = ||x - r(x)||^2.$$

An easy computation shows that the gradient of $\varphi$ is given by

$$\text{grad}\ \varphi = 2(x - r(x)).$$

Hence, for each point $x$ of the level surface $\partial N_\epsilon = \varphi^{-1}(\epsilon^2)$, the outward unit normal vector is given by

$$g(x) = \text{grad}\ \varphi / ||\text{grad}\ \varphi|| = (x - r(x))/\epsilon.$$

Extend $v$ to a vector field $w$ on the neighborhood $N_\epsilon$ by setting

$$w(x) = (x - r(x)) + v(r(x)).$$

Then $w$ points outward along the boundary, since the inner product $w(x) \cdot g(x)$ is equal to $\epsilon > 0$. Note that $w$ can vanish only at the zeros of $v$ in $M$; this is clear since the two summands $(x - r(x))$ and $v(r(x))$ are mutually orthogonal. Computing the derivative of $w$ at a zero $z \in M$, we see that

$$dw_z(h) = dv_z(h) \quad \text{for all} \quad h \in TM_z$$

$$dw_z(h) = h \quad \text{for} \quad h \in TM_z^\perp.$$

Thus the determinant of $dw_z$ is equal to the determinant of $dv_z$. Hence the index of $w$ at the zero $z$ is equal to the index $\iota$ of $v$ at $z$.

Now according to Lemma 3 the index sum $\sum \iota$ is equal to the degree of $g$. This proves Theorem 1.

Examples. On the sphere $S^m$ there exists a vector field $v$ which points "north" at every point.* At the south pole the vectors radiate outward; hence the index is $+1$. At the north pole the vectors converge inward; hence the index is $(-1)^m$. Thus the invariant $\sum \iota$ is equal to 0 or 2 according as $m$ is odd or even. This gives a new proof that every vector field on an even sphere has a zero.

For any odd-dimensional, boundaryless manifold the invariant $\sum \iota$ is zero. For if the vector field $v$ is replaced by $-v$, then each index is multiplied by $(-1)^m$, and the equality

$$\sum \iota = (-1)^m \sum \iota,$$

for $m$ odd, implies that $\sum \iota = 0$.

---

* For example, $v$ can be defined by the formula $v(x) = p - (p \cdot x)x$, where $p$ is the north pole. (See Figure 11.)

REMARK. If $\sum \iota = 0$ on a connected manifold $M$, then a theorem of Hopf asserts that there exists a vector field on $M$ with no zeros at all.

In order to obtain the full strength of the Poincare-Hopf theorem, three further steps are needed.

STEP 1. *Identification of the invariant $\sum \iota$ with the Euler number $\chi(M)$.* It is sufficient to construct just one example of a nondegenerate vector field on $M$ with $\sum \iota$ equal to $\chi(M)$. The most pleasant way of doing this is the following: According to Marston Morse, it is always possible to find a real valued function on $M$ whose "gradient" is a nondegenerate vector field. Furthermore, Morse showed that the sum of indices associated with such a gradient field is equal to the Euler number of $M$. For details of this argument the reader is referred to Milnor [22, pp. 29, 36].

STEP 2. *Proving the theorem for a vector field with degenerate zeros.* Consider first a vector field $v$ on an open set $U$ with an isolated zero at $z$. If

$$\lambda : U \rightarrow [0, 1]$$

takes the value 1 on a small neighborhood $N_1$ of $z$ and the value 0 outside a slightly larger neighborhood $N$, and if $y$ is a sufficiently small regular value of $v$, then the vector field

$$v'(x) = v(x) - \lambda(x)y$$

is nondegenerate* within $N$. The sum of the indices at the zeros within $N$ can be evaluated as the degree of the map

$$\bar{v} : \partial N \rightarrow S^{m-1},$$

and hence does not change during this alteration.

More generally consider vector fields on a compact manifold $M$. Applying this argument locally we see that *any vector field with isolated zeros can be replaced by a nondegenerate vector field without altering the integer $\sum \iota$.*

STEP 3. *Manifolds with boundary.* If $M \subset R^k$ has a boundary, then any vector field $v$ which points outward along $\partial M$ can again be extended over the neighborhood $N_\epsilon$ so as to point outward along $\partial N_\epsilon$. However, there is some difficulty with smoothness around the boundary of $M$. Thus $N_\epsilon$ is not a smooth (i.e. differentiable of class $C^\infty$) manifold,

* Clearly $v'$ is nondegenerate within $N_1$. But if $y$ is sufficiently small, then $v'$ will have no zeros at all within $N - N_1$.

but only a $C^1$-manifold. The extension $w$, if defined as before by $w(x) = v(r(x)) + x - r(x)$, will only be a continuous vector field near $\partial M$. The argument can nonetheless be carried out either by showing that our strong differentiability assumptions are not really necessary or by other methods.

# §7. FRAMED COBORDISM
# THE PONTRYAGIN CONSTRUCTION

THE degree of a mapping $M \to M'$ is defined only when the manifolds $M$ and $M'$ are oriented and have the same dimension. We will study a generalization, due to Pontryagin, which is defined for a smooth map

$$f : M \to S^p$$

from an arbitrary compact, boundaryless manifold to a sphere. First some definitions.

Let $N$ and $N'$ be compact $n$-dimensional submanifolds of $M$ with $\partial N = \partial N' = \partial M = \emptyset$. The difference of dimensions $m - n$ is called the *codimension* of the submanifolds.

DEFINITION. $N$ is *cobordant to $N'$ within $M$* if the subset

$$N \times [0, \epsilon) \cup N' \times (1 - \epsilon, 1]$$

of $M \times [0, 1]$ can be extended to a compact manifold

$$X \subset M \times [0, 1]$$

so that

$$\partial X = N \times 0 \cup N' \times 1,$$

and so that $X$ does not intersect $M \times 0 \cup M \times 1$ except at the points of $\partial X$.

Clearly cobordism is an equivalence relation. (See Figure 15.)

DEFINITION. A *framing* of the submanifold $N \subset M$ is a smooth function $\mathfrak{v}$ which assigns to each $x \in N$ a basis

$$\mathfrak{v}(x) = (v^1(x), \cdots, v^{m-n}(x))$$

for the space $TN_x^{\perp} \subset TM_x$ of normal vectors to $N$ in $M$ at $x$. (See

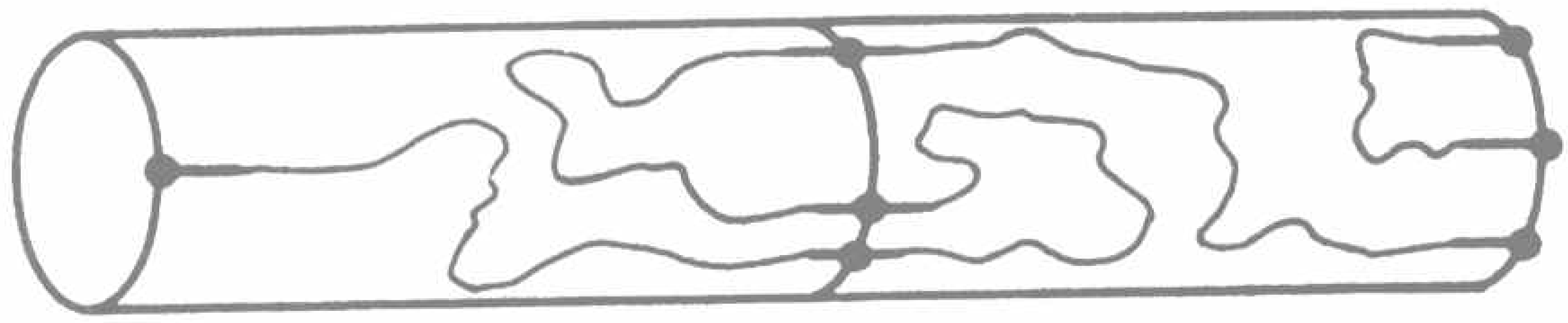


*Figure 15. Pasting together two cobordisms within M*

Figure 16.) The pair $(N, \mathfrak{v})$ is called a *framed submanifold* of $M$. Two framed submanifolds $(N, \mathfrak{v})$ and $(N', \mathfrak{w})$ are *framed cobordant* if there exists a cobordism $X \subset M \times [0, 1]$ between $N$ and $N'$ and a framing $\mathfrak{u}$ of $X$, so that

$$u^i(x, t) = (v^i(x), 0) \quad \text{for} \quad (x, t) \in N \times [0, \epsilon)$$
$$u^i(x, t) = (w^i(x), 0) \quad \text{for} \quad (x, t) \in N' \times (1 - \epsilon, 1].$$

Again this is an equivalence relation.

Now consider a smooth map $f : M \to S^p$ and a regular value $y \in S^p$. The map $f$ induces a framing of the manifold $f^{-1}(y)$ as follows: Choose a positively oriented basis $\mathfrak{v} = (v^1, \cdots, v^p)$ for the tangent space $T(S^p)_y$. For each $x \in f^{-1}(y)$ recall from page 12 that

$$df_x : TM_x \to T(S^p)_y$$

maps the subspace $Tf^{-1}(y)_x$ to zero and maps its orthogonal complement $Tf^{-1}(y)_x^{\perp}$ isomorphically onto $T(S^p)_y$. Hence there is a unique vector

$$w^i(x) \in Tf^{-1}(y)_x^{\perp} \subset TM_x$$

that maps into $v^i$ under $df_x$. It will be convenient to use the notation $\mathfrak{w} = f^*\mathfrak{v}$ for the resulting framing $w^1(x), \cdots, w^p(x)$ of $f^{-1}(y)$.

DEFINITION. This framed manifold $(f^{-1}(y), f^*\mathfrak{v})$ will be called the *Pontryagin manifold* associated with $f$.

Of course $f$ has many Pontryagin manifolds, corresponding to different choices of $y$ and $\mathfrak{v}$, but they all belong to a single framed cobordism class:

**Theorem A.** *If $y'$ is another regular value of $f$ and $\mathfrak{v}'$ is a positively oriented basis for $T(S^p)_{y'}$, then the framed manifold $(f^{-1}(y'), f^*\mathfrak{v}')$ is framed cobordant to $(f^{-1}(y), f^*\mathfrak{v})$.*

**Theorem B.** *Two mappings from $M$ to $S^p$ are smoothly homotopic if and only if the associated Pontryagin manifolds are framed cobordant.*

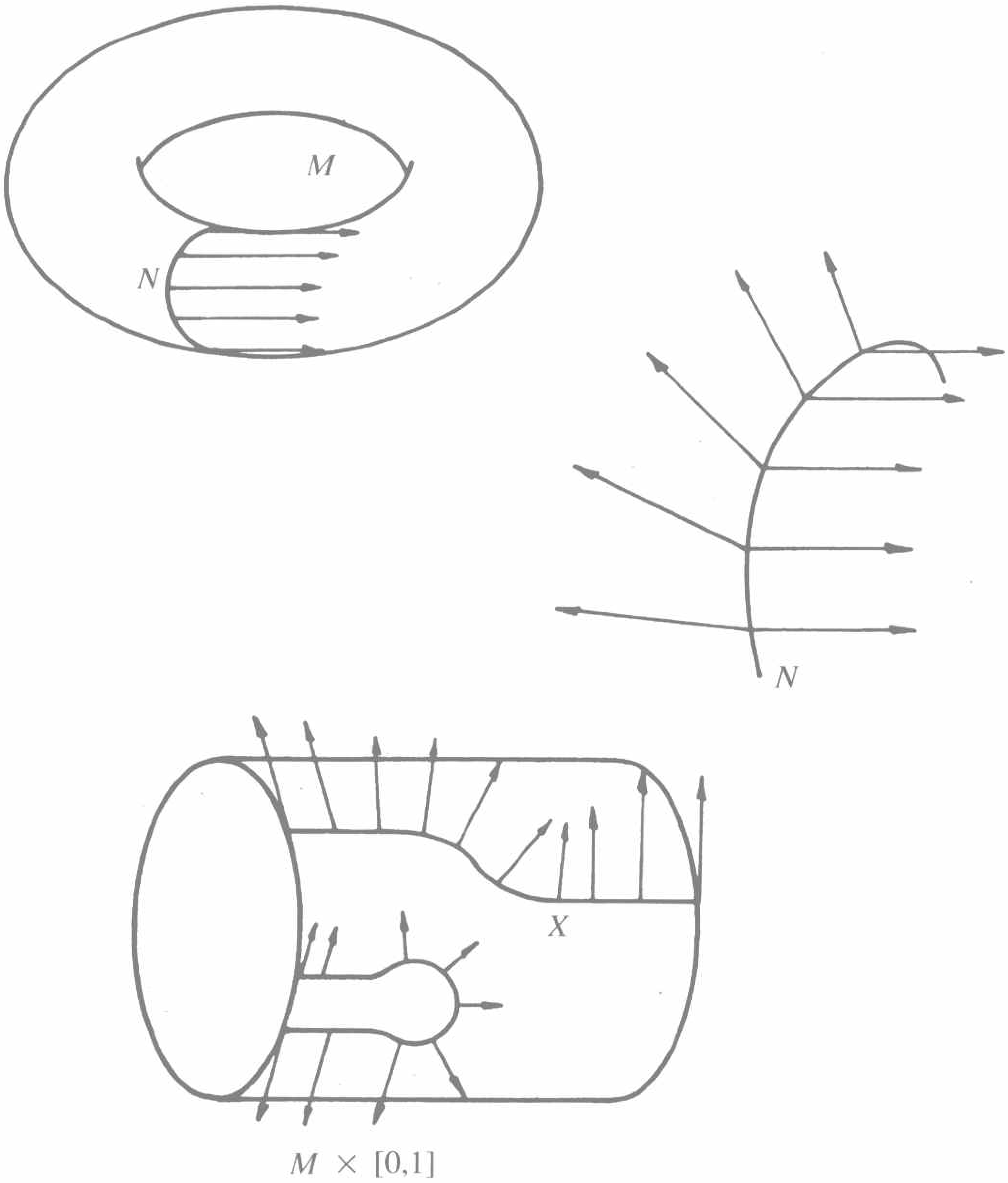


*Figure 16. Framed submanifolds and a framed cobordism*

**Theorem C.** *Any compact framed submanifold $(N, \mathfrak{w})$ of codimension $p$ in $M$ occurs as Pontryagin manifold for some smooth mapping $f : M \to S^p$.*

Thus the homotopy classes of maps are in one-one correspondence with the framed cobordism classes of submanifolds.

The proof of Theorem A will be very similar to the arguments in §§4 and 5. It will be based on three lemmas.

**Lemma 1.** *If $\mathfrak{v}$ and $\mathfrak{v}'$ are two different positively oriented bases at $y$, then the Pontryagin manifold $(f^{-1}(y), f^*\mathfrak{v})$ is framed cobordant to $(f^{-1}(y), f^*\mathfrak{v}')$.*

Proof. Choose a smooth path from $\mathfrak{v}$ to $\mathfrak{v}'$ in the space of all positively oriented bases for $T(S^p)_y$. This is possible since this space of bases

can be identified with the space $GL^+(p, R)$ of matrices with positive determinant, and hence is connected. Such a path gives rise to the required framing of the cobordism $f^{-1}(y) \times [0, 1]$.

By abuse of notation we will often delete reference to $f^*\mathfrak{v}$ and speak simply of "the framed manifold $f^{-1}(y)$."

**Lemma 2.** *If $y$ is a regular value of $f$, and $z$ is sufficiently close to $y$, then $f^{-1}(z)$ is framed cobordant to $f^{-1}(y)$.*

PROOF. Since the set $f(C)$ of critical values is compact, we can choose $\epsilon > 0$ so that the $\epsilon$-neighborhood of $y$ contains only regular values. Given $z$ with $\|z - y\| < \epsilon$, choose a smooth one-parameter family of rotations (i.e. an isotopy) $r_t : S^p \to S^p$ so that $r_1(y) = z$, and so that

1) $r_t$ is the identity for $0 \leq t < \epsilon'$,
2) $r_t$ equals $r_1$ for $1 - \epsilon' < t \leq 1$, and
3) each $r_t^{-1}(z)$ lies on the great circle from $y$ to $z$, and hence is a regular value of $f$.

Define the homotopy

$$F : M \times [0, 1] \to S^p$$

by $F(x, t) = r_t f(x)$. For each $t$ note that $z$ is a regular value of the composition

$$r_t \circ f : M \to S^p.$$

It follows a fortiori that $z$ is a regular value for the mapping $F$. Hence

$$F^{-1}(z) \subset M \times [0, 1]$$

is a framed manifold and provides a framed cobordism between the framed manifolds $f^{-1}(z)$ and $(r_1 \circ f)^{-1}(z) = f^{-1}r_1^{-1}(z) = f^{-1}(y)$. This proves Lemma 2.

**Lemma 3.** *If $f$ and $g$ are smoothly homotopic and $y$ is a regular value for both, then $f^{-1}(y)$ is framed cobordant to $g^{-1}(y)$.*

PROOF. Choose a homotopy $F$ with

$$F(x, t) = f(x) \qquad 0 \leq t < \epsilon,$$
$$F(x, t) = g(x) \qquad 1 - \epsilon < t \leq 1.$$

Choose a regular value $z$ for $F$ which is close enough to $y$ so that $f^{-1}(z)$ is framed cobordant to $f^{-1}(y)$ and so that $g^{-1}(z)$ is framed cobordant to $g^{-1}(y)$. Then $F^{-1}(z)$ is a framed manifold and provides a framed cobordism between $f^{-1}(z)$ and $g^{-1}(z)$. This proves Lemma 3.

PROOF OF THEOREM A. Given any two regular values $y$ and $z$ for $f$, we can choose rotations

$$r_t : S^p \to S^p$$

so that $r_0$ is the identity and $r_1(y) = z$. Thus $f$ is homotopic to $r_1 \circ f$; hence $f^{-1}(z)$ is framed cobordant to

$$(r_1 \circ f)^{-1}(z) = f^{-1}r_1^{-1}(z) = f^{-1}(y).$$

This completes the proof of Theorem A.

The proof of Theorem C will be based on the following: Let $N \subset M$ be a framed submanifold of codimension $p$ with framing $\mathfrak{v}$. Assume that $N$ is compact and that $\partial N = \partial M = \varnothing$.

**Product Neighborhood Theorem.** *Some neighborhood of $N$ in $M$ is diffeomorphic to the product $N \times R^p$. Furthermore the diffeomorphism can be chosen so that each $x \in N$ corresponds to $(x, 0) \in N \times R^p$ and so that each normal frame $\mathfrak{v}(x)$ corresponds to the standard basis for $R^p$.*

REMARK. Product neighborhoods do not exist for arbitrary submanifolds. (Compare Figure 17.)

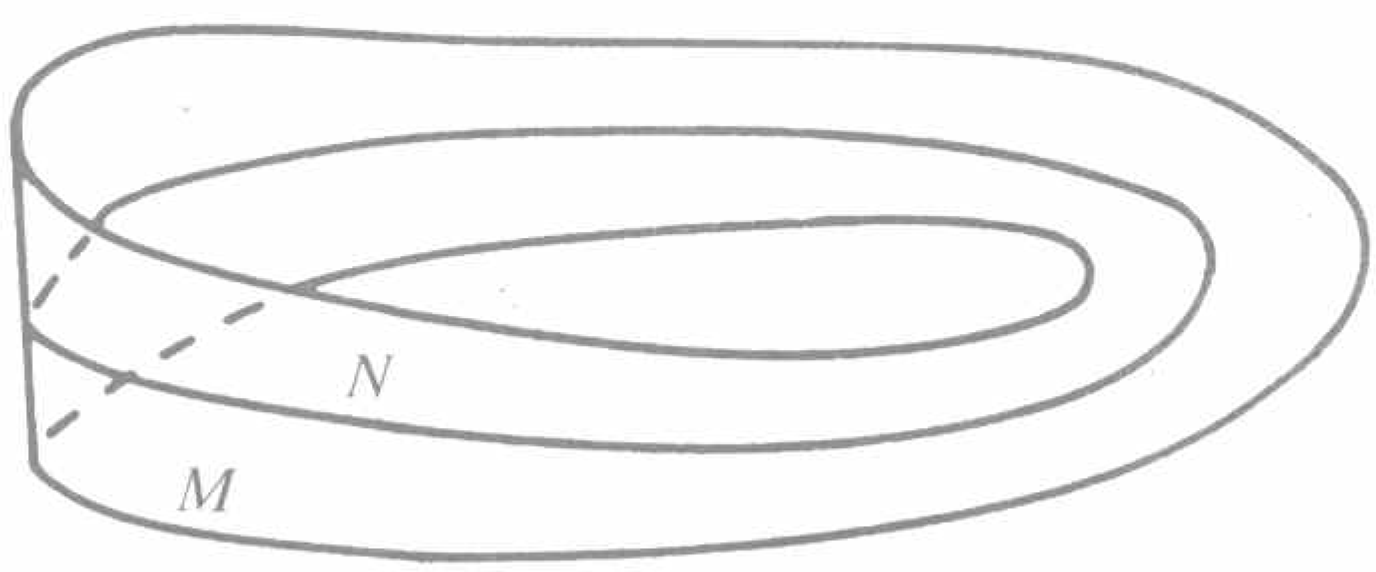


*Figure 17. An unframable submanifold*

PROOF. First suppose that $M$ is the euclidean space $R^{n+p}$. Consider the mapping $g : N \times R^p \to M$, defined by

$$g(x; t_1, \cdots, t_p) = x + t_1v^1(x) + \cdots + t_pv^p(x).$$

Clearly $dg_{(x;0,\cdots,0)}$ is nonsingular; hence $g$ maps some neighborhood of $(x, 0) \in N \times R^p$ diffeomorphically onto an open set.

We will prove that $g$ is one-one on the entire neighborhood $N \times U_\epsilon$ of $N \times 0$, providing that $\epsilon > 0$ is sufficiently small; where $U_\epsilon$ denotes the $\epsilon$-neighborhood of 0 in $R^p$. For otherwise there would exist pairs $(x, u) \neq (x', u')$ in $N \times R^p$ with $||u||$ and $||u'||$ arbitrarily small and with

$$g(x, u) = g(x', u').$$

Since $N$ is compact, we could choose a sequence of such pairs with $x$ converging, say to $x_0$, with $x'$ converging to $x_0'$, and with $u \to 0$ and $u' \to 0$. Then clearly $x_0 = x_0'$, and we have contradicted the statement that $g$ is one-one in a neighborhood of $(x_0, 0)$.

Thus $g$ maps $N \times U_\epsilon$ diffeomorphically onto an open set. But $U_\epsilon$ is diffeomorphic to the full euclidean space $R^p$ under the correspondence

$$u \to u/(1 - ||u||^2/\epsilon^2).$$

Since $g(x, 0) = x$, and since $dg_{(x,0)}$ does what is expected of it, this proves the Product Neighborhood Theorem for the special case $M = R^{n+p}$.

For the general case it is necessary to replace straight lines in $R^{n+p}$ by geodesics in $M$. More precisely let $g(x; t_1, \cdots, t_p)$ be the endpoint of the geodesic segment of length $||t_1v^1(x) + \cdots + t_pv^p(x)||$ in $M$ which starts at $x$ with the initial velocity vector

$$t_1v^1(x) + \cdots + t_pv^p(x)/||t_1v^1(x) + \cdots + t_pv^p(x)||.$$

The reader who is familiar with geodesics will have no difficulty in checking that

$$g : N \times U_\epsilon \to M$$

is well defined and smooth, for $\epsilon$ sufficiently small. The remainder of the proof proceeds exactly as before.

Proof of Theorem C. Let $N \subset M$ be a compact, boundaryless, framed submanifold. Choose a product representation

$$g : N \times R^p \to V \subset M$$

for a neighborhood $V$ of $N$, as above, and define the projection

$$\pi : V \to R^p$$

by $\pi(g(x, y)) = y$. (See Figure 18.) Clearly 0 is a regular value, and $\pi^{-1}(0)$ is precisely $N$ with its given framing.

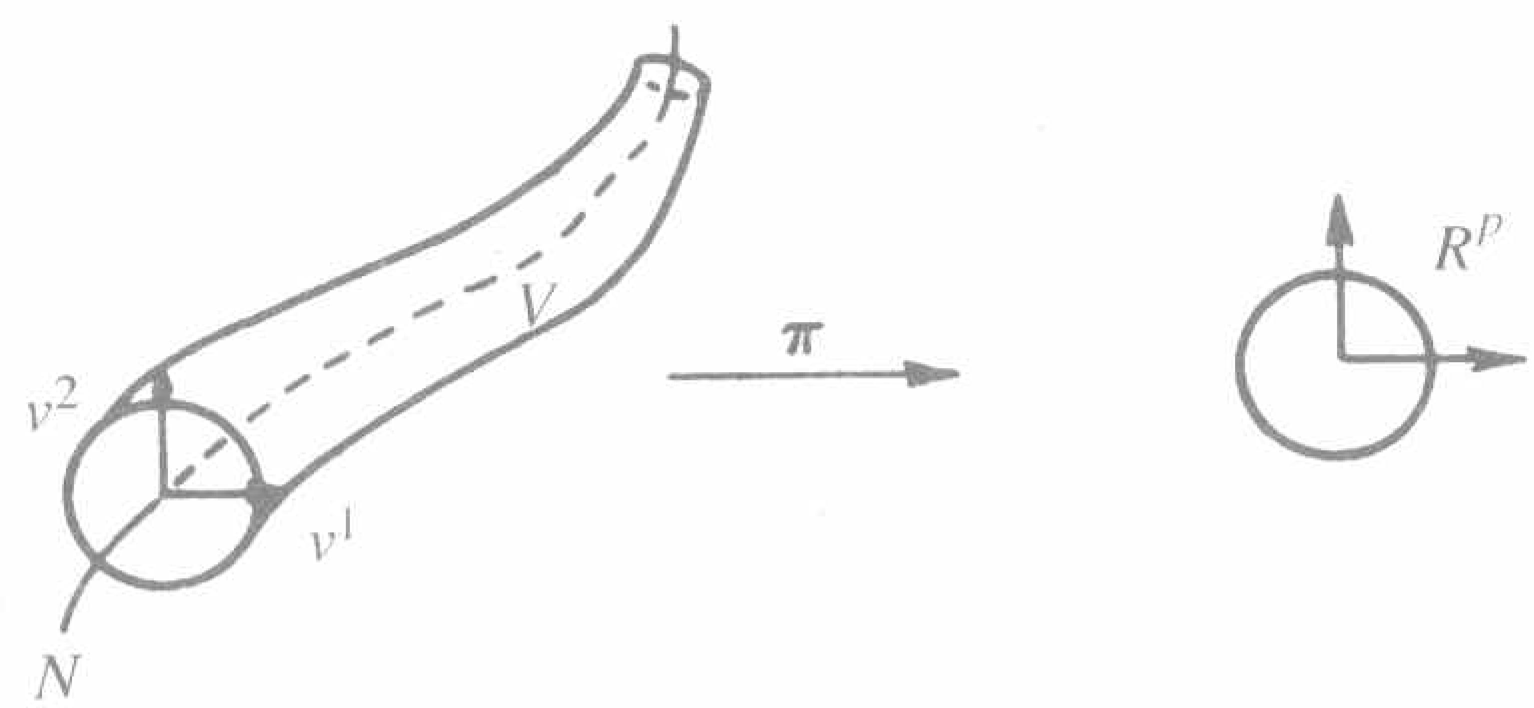


*Figure 18. Constructing a map with given Pontryagin manifold*

Now choose a smooth map $\varphi : R^p \to S^p$ which maps every $x$ with $||x|| \geq 1$ into a base point $s_0$, and maps the open unit ball in $R^p$ diffeomorphically* onto $S^p - s_0$. Define

$$f : M \to S^p$$

by

$$f(x) = \varphi(\pi(x)) \quad \text{for} \quad x \in V$$

$$f(x) = s_0 \qquad \text{for} \quad x \notin V.$$

Clearly $f$ is smooth, and the point $\varphi(0)$ is a regular value of $f$. Since the corresponding Pontryagin manifold

$$f^{-1}(\varphi(0)) = \pi^{-1}(0)$$

is precisely equal to the framed manifold $N$, this completes the proof of Theorem C.

In order to prove Theorem B we must first show that the Pontryagin manifold of a map determines its homotopy class. Let $f,\ g : M \to S^p$ be smooth maps with a common regular value $y$.

**Lemma 4.** *If the framed manifold* $(f^{-1}(y), f^*\mathfrak{v})$ *is equal to* $(g^{-1}(y), g^*\mathfrak{v})$, *then $f$ is smoothly homotopic to $g$.*

Proof. It will be convenient to set $N = f^{-1}(y)$. The hypothesis that $f^*\mathfrak{v} = g^*\mathfrak{v}$ means that $df_x = dg_x$ for all $x \in N$.

First suppose that $f$ actually coincides with $g$ throughout an entire neighborhood $V$ of $N$. Let $h : S^p - y \to R^p$ be stereographic projection. Then the homotopy

$$H(x, t) = f(x) \qquad\qquad \text{for} \quad x \in V$$

$$H(x, t) = h^{-1}[t \cdot h(f(x)) + (1 - t) \cdot h(g(x))] \quad \text{for} \quad x \in M - N$$

proves that $f$ is smoothly homotopic to $g$.

Thus is suffices to deform $f$ so that it coincides with $g$ in some small neighborhood of $N$, being careful not to map any new points into $y$ during the deformation. Choose a product representation

$$N \times R^p \to V \subset M$$

for a neighborhood $V$ of $N$, where $V$ is small enough so that $f(V)$ and

* For example, $\varphi(x) = h^{-1}(x/\lambda(||x||^2))$, where $h$ is the stereographic projection from $s_0$ and where $\lambda$ is a smooth monotone decreasing function with $\lambda(t) > 0$ for $t < 1$ and $\lambda(t) = 0$ for $t \geq 1$.

$g(V)$ do not contain the antipode $\bar{y}$ of $y$. Identifying $V$ with $N \times R^p$ and identifying $S^p - \bar{y}$ with $R^p$, we obtain corresponding mappings

$$F, G : N \times R^p \to R^p,$$

with

$$F^{-1}(0) = G^{-1}(0) = N \times 0,$$

and with

$$dF_{(x,0)} = dG_{(x,0)} = (\text{projection to } R^p)$$

for all $x \in N$.

We will first find a constant $c$ so that

$$F(x, u)\cdot u > 0, \qquad G(x, u)\cdot u > 0$$

for $x \in N$ and $0 < ||u|| < c$. That is, the points $F(x, u)$ and $G(x, u)$ belong to the same open half-space in $R^p$. So the homotopy

$$(1 - t)F(x, u) + tG(x, u)$$

between $F$ and $G$ will not map any new points into 0, at least for $||u|| < c$.

By Taylor's theorem

$$||F(x, u) - u|| \leq c_1 \, ||u||^2, \quad \text{for} \quad ||u|| \leq 1.$$

Hence

$$|(F(x, u) - u)\cdot u| \leq c_1 \, ||u||^3$$

and

$$F(x, u)\cdot u \geq ||u||^2 - c_1 \, ||u||^3 > 0$$

for $0 < ||u|| < c = \text{Min}\,(c_1^{-1}, 1)$, with a similar inequality for $G$.

To avoid moving distant points we select a smooth map $\lambda : R^p \to R$ with

$$\lambda(u) = 1 \quad \text{for} \quad ||u|| \leq c/2$$

$$\lambda(u) = 0 \quad \text{for} \quad ||u|| \geq c.$$

Now the homotopy

$$F_t(x, u) = [1 - \lambda(u)t]F(x, u) + \lambda(u)tG(x, u)$$

deforms $F = F_0$ into a mapping $F_1$ that (1) coincides with $G$ in the region $||u|| < c/2$, (2) coincides with $F$ for $||u|| \geq c$, and (3) has no new zeros. Making a corresponding deformation of the original mapping $f$, this clearly completes the proof of Lemma 4.

Proof of Theorem B. If $f$ and $g$ are smoothly homotopic, then Lemma 3 asserts that the Pontryagin manifolds $f^{-1}(y)$ and $g^{-1}(y)$ are framed cobordant. Conversely, given a framed cobordism $(X, \mathfrak{w})$ between $f^{-1}(y)$ and $g^{-1}(y)$, an argument completely analogous to the proof of Theorem C constructs a homotopy

$$F : M \times [0, 1] \to S^p$$

whose Pontryagin manifold $(F^{-1}(y), F^*\mathfrak{v})$ is precisely equal to $(X, \mathfrak{w})$. Setting $F_t(x) = F(x, t)$, note that the maps $F_0$ and $f$ have exactly the same Pontryagin manifold. Hence $F_0 \sim f$ by Lemma 4; and similarly $F_1 \sim g$. Therefore $f \sim g$, which completes the proof of Theorem B.

Remarks. Theorems A, B, and C can easily be generalized so as to apply to a manifold $M$ with boundary. The essential idea is to consider only mappings which carry the boundary into a base point $s_0$. The homotopy classes of such mappings

$$(M, \partial M) \to (S^p, s_0)$$

are in one-one correspondence with the cobordism classes of framed submanifolds

$$N \subset \text{Interior}(M)$$

of codimension $p$. If $p \geq \frac{1}{2}m + 1$, then this set of homotopy classes can be given the structure of an abelian group, called the $p$-th *cohomotopy group* $\pi^p(M, \partial M)$. The composition operation in $\pi^p(M, \partial M)$ corresponds to the union operation for disjoint framed submanifolds of Interior $(M)$. (Compare §8, Problem 17.)

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## THE HOPF THEOREM

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As an example, let $M$ be a connected and oriented manifold of dimension $m = p$. A framed submanifold of codimension $p$ is just a finite set of points with a preferred basis at each. Let $\operatorname{sgn}(x)$ equal $+1$ or $-1$ according as the preferred basis determines the right or wrong orientation. Then $\sum \operatorname{sgn}(x)$ is clearly equal to the degree of the associated map $M \to S^m$. But it is not difficult to see that the framed cobordism class of the 0-manifold is uniquely determined by this integer $\sum \operatorname{sgn}(x)$. Thus we have proved the following.

**Theorem of Hopf.** *If $M$ is connected, oriented, and boundaryless, then two maps $M \to S^m$ are smoothly homotopic if and only if they have the same degree.*

On the other hand, suppose that $M$ is not orientable. Then given a basis for $TM_x$ we can slide $x$ around $M$ in a closed loop so as to transform the given basis into one of opposite orientation. An easy argument then proves the following:

**Theorem.** *If $M$ is connected but nonorientable, then two maps $M \to S^m$ are homotopic if and only if they have the same mod 2 degree.*

The theory of framed cobordism was introduced by Pontryagin in order to study homotopy classes of mappings

$$S^m \to S^p$$

with $m > p$. For example if $m = p + 1 \geq 4$, there are precisely two homotopy classes of mappings $S^m \to S^p$. Pontryagin proved this result by classifying framed 1-manifolds in $S^m$. With considerably more difficulty he was able to show that there are just two homotopy classes also in the case $m = p + 2 \geq 4$, using framed 2-manifolds. However, for $m - p > 2$ this approach to the problem runs into manifold difficulties.

It has since turned out to be easier to enumerate homotopy classes of mappings by quite different, more algebraic methods.* Pontryagin's construction is, however, a double-edged tool. It not only allows us to translate information about manifolds into homotopy theory; it conversely enables us to translate any information about homotopy into manifold theory. Some of the deepest work in modern topology has come from the interplay of these two theories. Rene Thom's work on cobordism is an important example of this. (References [36], [21].)

* See for example S.-T. Hu, *Homotopy Theory*.

# §8. EXERCISES

IN CONCLUSION here are some problems for the reader.

PROBLEM 1. Show that the degree of a composition $g \circ f$ is equal to the product (degree $g$)(degree $f$).

PROBLEM 2. Show that every complex polynomial of degree $n$ gives rise to a smooth map from the Gauss sphere $S^2$ to itself of degree $n$.

PROBLEM 3. If two maps $f$ and $g$ from $X$ to $S^p$ satisfy $\|f(x) - g(x)\| < 2$ for all $x$, prove that $f$ is homotopic to $g$, the homotopy being smooth if $f$ and $g$ are smooth.

PROBLEM 4. If $X$ is compact, show that every continuous map $X \to S^p$ can be uniformly approximated by a smooth map. If two smooth maps $X \to S^p$ are continuously homotopic, show that they are smoothly homotopic.

PROBLEM 5. If $m < p$, show that every map $M^m \to S^p$ is homotopic to a constant.

PROBLEM 6. (Brouwer). Show that any map $S^n \to S^n$ with degree different from $(-1)^{n+1}$ must have a fixed point.

PROBLEM 7. Show that any map $S^n \to S^n$ of odd degree must carry some pair of antipodal points into a pair of antipodal points.

PROBLEM 8. Given smooth manifolds $M \subset R^k$ and $N \subset R^l$, show that the tangent space $T(M \times N)_{(x,y)}$ is equal to $TM_x \times TN_y$.

PROBLEM 9. The graph $\Gamma$ of a smooth map $f : M \to N$ is defined to be the set of all $(x, y) \in M \times N$ with $f(x) = y$. Show that $\Gamma$ is a smooth

manifold and that the tangent space

$$T\Gamma_{(x,y)} \subset TM_x \times TN_y$$

is equal to the graph of the linear map $df_x$.

PROBLEM 10. Given $M \subset R^k$, show that the *tangent bundle space*

$$TM = \{(x, v) \in M \times R^k \mid v \in TM_x\}$$

is also a smooth manifold. Show that any smooth map $f : M \to N$ gives rise to a smooth map

$$df : TM \to TN$$

where

$$d(\text{identity}) = \text{identity}, \quad d(g \circ f) = (dg) \circ (df).$$

PROBLEM 11. Similarly show that the *normal bundle space*

$$E = \{(x, v) \in M \times R^k \mid v \perp TM_x\}$$

is a smooth manifold. If $M$ is compact and boundaryless, show that the correspondence

$$(x, v) \mapsto x + v$$

from $E$ to $R^k$ maps the $\epsilon$-neighborhood of $M \times 0$ in $E$ diffeomorphically onto the $\epsilon$-neighborhood $N_\epsilon$ of $M$ in $R^k$. (Compare the Product Neighborhood Theorem in §7.)

PROBLEM 12. Define $r : N_\epsilon \to M$ by $r(x + v) = x$. Show that $r(x + v)$ is closer to $x + v$ than any other point of $M$. Using this retraction $r$, prove the analogue of Problem 4 in which the sphere $S^p$ is replaced by a manifold $M$.

PROBLEM 13. Given disjoint manifolds $M, N \subset R^{k+1}$, the linking map

$$\lambda : M \times N \to S^k$$

is defined by $\lambda(x, y) = (x - y)/\|x - y\|$. If $M$ and $N$ are compact, oriented, and boundaryless, with total dimension $m + n = k$, then the degree of $\lambda$ is called the *linking number* $l(M, N)$. Prove that

$$l(N, M) = (-1)^{(m+1)(n+1)} l(M, N).$$

If $M$ bounds an oriented manifold $X$ disjoint from $N$, prove that $l(M, N) = 0$. Define the linking number for disjoint manifolds in the sphere $S^{m+n+1}$

Problem 14, The Hopf Invariant. If $y \neq z$ are regular values for a map $f : S^{2p-1} \rightarrow S^p$, then the manifolds $f^{-1}(y)$, $f^{-1}(z)$ can be oriented as in §5; hence the linking number $l(f^{-1}(y), f^{-1}(z))$ is defined.

a) Prove that this linking number is locally constant as a function of $y$.

b) If $y$ and $z$ are regular values of $g$ also, where

$$\|f(x) - g(x)\| < \|y - z\|$$

for all $x$, prove that

$$l(f^{-1}(y), f^{-1}(z)) = l(g^{-1}(y), f^{-1}(z)) = l(g^{-1}(y), g^{-1}(z)).$$

c) Prove that $l(f^{-1}(y), f^{-1}(z))$ depends only on the homotopy class of $f$, and does not depend on the choice of $y$ and $z$.

This integer $H(f) = l(f^{-1}(y), f^{-1}(z))$ is called the *Hopf invariant* of $f$. (Reference [15].)

Problem 15. If the dimension $p$ is odd, prove that $H(f) = 0$. For a composition

$$S^{2p-1} \xrightarrow{f} S^p \xrightarrow{g} S^p$$

prove that $H(g \circ f)$ is equal to $H(f)$ multiplied by the square of the degree of $g$.

The *Hopf fibration* $\pi : S^3 \rightarrow S^2$ is defined by

$$\pi(x_1, x_2, x_3, x_4) = h^{-1}((x_1 + ix_2)/(x_3 + ix_4))$$

where $h$ denotes stereographic projection to the complex plane. Prove that $H(\pi) = 1$.

Problem 16. Two submanifolds $N$ and $N'$ of $M$ are said to *intersect transversally* if, for each $x \in N \cap N'$, the subspaces $TN_x$ and $TN'_x$ together generate $TM_x$. (If $n + n' < m$ this means that $N \cap N' = \emptyset$.) If $N$ is a framed submanifold, prove that it can be deformed slightly so as to intersect a given $N'$ transversally. Prove that the resulting intersection is a smooth manifold.

Problem 17. Let $\Pi^p(M)$ denote the set of all framed cobordism classes of codimension $p$ in $M$. Use the transverse intersection operation to define a correspondence

$$\Pi^p(M) \times \Pi^q(M) \rightarrow \Pi^{p+q}(M).$$

If $p \geq \frac{1}{2}m + 1$, use the disjoint union operation to make $\Pi^p(M)$ into an abelian group. (Compare p. 50.)

# APPENDIX

# CLASSIFYING 1-MANIFOLDS

WE WILL prove the following result, which has been assumed in the text. A brief discussion of the classification problem for higher dimensional manifolds will also be given.

**Theorem.** *Any smooth, connected 1-dimensional manifold is diffeomorphic either to the circle $S^1$ or to some interval of real numbers.*

(An *interval* is a connected subset of $R$ which is not a point. It may be finite or infinite; closed, open, or half-open.)

Since any interval is diffeomorphic* either to $[0, 1]$, $(0, 1]$, or $(0, 1)$, it follows that there are only four distinct connected 1-manifolds.

The proof will make use of the concept of "arc-length." Let $I$ be an interval.

DEFINITION. A map $f : I \to M$ is a *parametrization by arc-length* if $f$ maps $I$ diffeomorphically onto an open subset† of $M$, and if the "velocity vector" $df_s(1) \in TM_{f(s)}$ has unit length, for each $s \in I$.

Any given local parametrization $I' \to M$ can be transformed into a parametrization by arc-length by a straightforward change of variables.

**Lemma.** *Let $f : I \to M$ and $g : J \to M$ be parametrizations by arc-length. Then $f(I) \cap g(J)$ has at most two components. If it has only one component, then $f$ can be extended to a parametrization by arc-length of the union $f(I) \cup g(J)$. If it has two components, then $M$ must be diffeomorphic to $S^1$.*

---

* For example, use a diffeomorphism of the form

$$f(t) = a \tanh (t) + b.$$

† Thus $I$ can have boundary points only if $M$ has boundary points.

PROOF. Clearly $g^{-1} \circ f$ maps some relatively open subset of $I$ diffeomorphically onto a relatively open subset of $J$. Furthermore the derivative of $g^{-1} \circ f$ is equal to $\pm 1$ everywhere.

Consider the graph $\Gamma \subset I \times J$, consisting of all $(s, t)$ with $f(s) = g(t)$. Then $\Gamma$ is a closed subset of $I \times J$ made up of line segments of slope $\pm 1$. Since $\Gamma$ is closed and $g^{-1} \circ f$ is locally a diffeomorphism, these line segments cannot end in the interior of $I \times J$, but must extend to the boundary. Since $g^{-1} \circ f$ is one-one and single valued, there can be at most one of these segments ending on each of the four edges of the rectangle $I \times J$. Hence $\Gamma$ has at most two components. (See Figure 19.) Furthermore, if there are two components, the two must have the same slope.

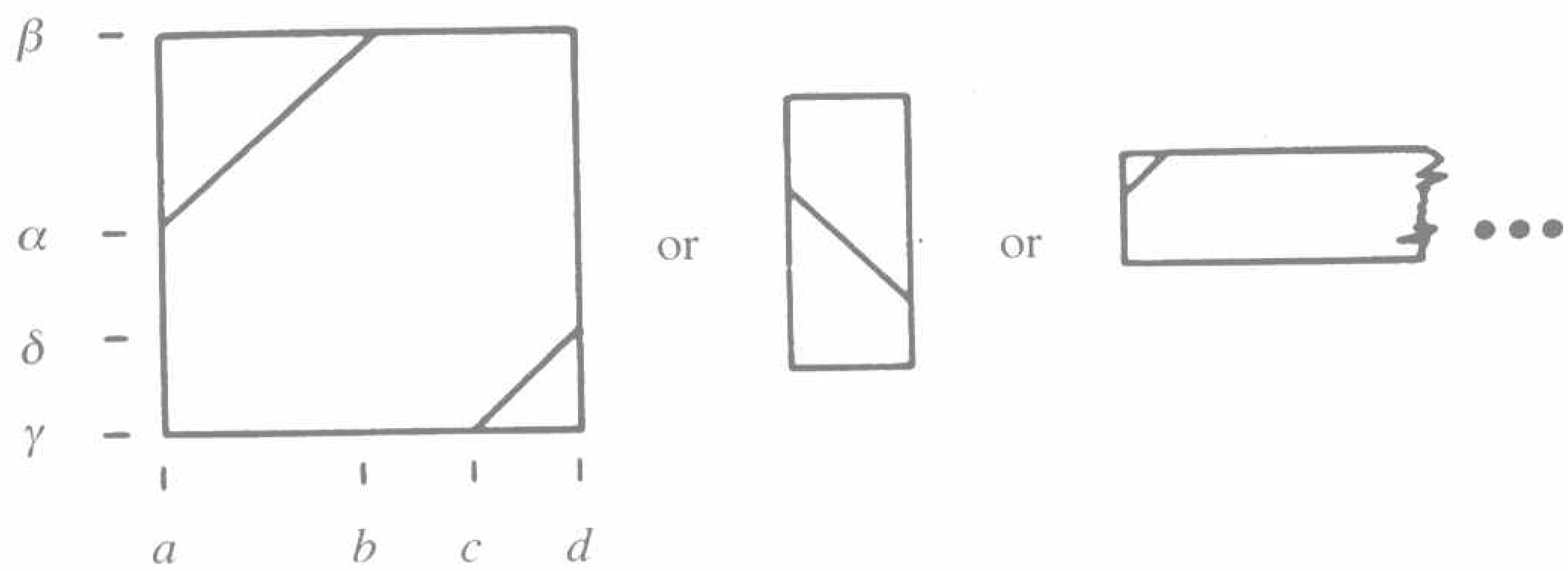


*Figure 19. Three of the possibilities for* $\Gamma$

If $\Gamma$ is connected, then $g^{-1} \circ f$ extends to a linear map $L : R \to R$. Now $f$ and $g \circ L$ piece together to yield the required extension

$$F : I \cup L^{-1}(J) \to f(I) \cup g(J).$$

If $\Gamma$ has two components, with slope say $+1$, they must be arranged as in the left-hand rectangle of Figure 19. Translating the interval $J = (\gamma, \beta)$ if necessary, we may assume that $\gamma = c$ and $\delta = d$, so that

$$a < b \leq c < d \leq \alpha < \beta.$$

Now setting $\theta = 2\pi t/(\alpha - a)$, the required diffeomorphism

$$h : S^1 \to M$$

is defined by the formula

$$\begin{aligned} h(\cos \theta, \sin \theta) &= f(t) \quad \text{for} \quad a < t < d, \\ &= g(t) \quad \text{for} \quad c < t < \beta. \end{aligned}$$

The image $h(S^1)$, being compact and open in $M$, must be the entire manifold $M$. This proves the lemma.

Proof of Classification Theorem. Any parametrization by arc-length can be extended to one

$$f : I \to M$$

which is maximal in the sense that $f$ cannot be extended over any larger interval as a parametrization by arc-length: it is only necessary to extend $f$ as far as possible to the left and then as far as possible to the right.

If $M$ is not diffeomorphic to $S^1$, we will prove that $f$ is onto, and hence is a diffeomorphism. For if the open set $f(I)$ were not all of $M$, there would be a limit point $x$ of $f(I)$ in $M - f(I)$. Parametrizing a neighborhood of $x$ by arc-length and applying the lemma, we would see that $f$ can be extended over a larger interval. This contradicts the assumption that $f$ is maximal and hence completes the proof.

Remarks. For manifolds of higher dimension the classification problem becomes quite formidable. For 2-dimensional manifolds, a thorough exposition has been given by Kerekjarto [17]. The study of 3-dimensional manifolds is very much a topic of current research. (See Papakyriakopoulos [26].) For compact manifolds of dimension $\geq 4$ the classification problem is actually unsolvable.* But for high dimensional simply connected manifolds there has been much progress in recent years, as exemplified by the work of Smale [31] and Wall [37].

* See Markov [19].

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